Measure And Integration

Seminar Integration (1963)

Two Courses by A. P. Morse

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Introduction

From the 1950's until his death in 1984, A. P. Morse made great strides in developing a usable formal language for mathematics. His aim was to have a completely formal language that was easy to use, often intuitive, and in agreement with normal informal practice whenever possible. His book, *A Theory of Sets* (Academic Press, 1965; 2nd ed. 1986), provides the most complete exposure of his ideas in publication. The book is sparse and, in accordance with its title, deals with set theory and its foundations only. In the book there are hints that the language is intended for use in higher mathematics, but without seeing how such a program might be carried out, it is difficult to appreciate the possibilities.

This publication demonstrates how Morse used his formal language in two courses that he taught at the University of California at Berkeley:

Measure and Integration
Seminar Integration.

We will refer to these courses as MI and SI, respectively. Both courses deal with the theory of integration and have a fair amount of material in common, but they show significantly different aspects of the subject. MI covers the real numbers, unordered real summation, measure theory, and real integration. SI, on the other hand, begins with the complex plane and develops unordered complex summation and complex integration, with very little mention of measures.

The purpose of this introduction is to provide additional explanation to assist the reader in understanding some of the notations and concepts and to provide insight into some of the results.

The Importance of Morse's Work

The idea that mathematics is governed by strict rules regarding both symtax and proof is the basis for the systems developed by Frege, Peano, Russell and Whitehead, Quine, Morse and others. Such rules have usually resulted in expressions that are difficult to code and read and proofs that are tedious and impractically time-consuming. In his book, Morse gives one complete proof in logic and says "We shall never again be so detailed in our proofs." However, while Morse did not see a way clear to making formal proofs practical, he did believe that formal expression of theorems was practical. His rules of syntax allow considerable flexibility for writing formal expressions and he showed how these rules could be exploited to make formal mathematics readable.

Even if one takes the effort to follow all the rules, what assurance is there that the result is correct? Morse's book contained several typographical errors, but undoubtably fewer than most technical books. His only assurance of correctness was through careful proofreading. Today, there is a better way to be assured of formal correctness, thanks to the power of the computer. Given a suitable way to encode into a computer language the formal images that are to appear on the written page, such as the $\TeX$ program of Knuth, a program can be written to read the coding and parse the formal expressions for correctness. Such a program, called ProofCheck, has been written in Python by Bob Neveln. ProofCheck also sythesizes the rules of proof laid out by Morse. The resulting rules of proof are much closer to being practical than Morse's original set of rules. Thus we can see Morse's work as the foundation for a practical computerized formal mathematics.

Morse's Style

Before getting into some of the specifics, it may be helpful to look at Morse's style of presentation. This style can be characterized as incremental development supplemented by occassional proofs. His preference is to state many "Theorems", most of which are such small steps from previously known results that the reader is expected to easily supply the proof. The use of many smaller theorems is similar to the style employed by Edmund Landau (cf. *Foundations of Analysis*). Sometimes hints are given for proofs and, when needed, more detailed proofs are provided. To get a sense of this incremental development, we give the number of theorems (and occassional lemmas) in each of several sections of SI.

Topic	Definitions	Postulates	Theorems
Number systems	38	28	119
Limits	34	0	41
Finite unordered summation	6	0	27
General unordered infinite summation	7	0	48
Restricted unordered infinite summation	8	0	91

Morse's Logic

Morse adopted the predominant view that all mathematics is to be developed in set theory. This view owes much to Dedekind who showed how to construct the number systems out of sets and formulated a set-theoretic foundation for general induction. Set theory is a logical theory, that is to say that there is first a foundation of sentential and predicate logic upon which set theory sits. Morse's logic (and the resulting set theory) is unusual in that no differentiation is made between terms (object names) and formulas (statement names). However strange this may appear, in practice it has only marginal bearing on the formal statements in these notes. Later on, we discuss a few instances where this may be perplexing.

The sentential logic is standard in that the true theorems are the tautologies. The notation is also standard as shown below where p and q are statements.

Notation	Reading	Alternate Reading	Truth Condition
0	false		false
U	true		true
$(p \to q)$	p implies q	If p then p	p is false or q is true
$(p \wedge q)$	p and q		Both p and q are true
$(p \vee q)$	p or q		p is true or q is true, or both
$(p \leftrightarrow q)$	p if and only if q	p is equivalent to q	both or true or both are false
$\sim p$	not p		p is false

In more complex expressions, removal of parentheses is governed by the precedence of operators. A table of such precedences appears later in this introduction. For example, the expression

$$((p \wedge q) \to (p \vee q))$$

may be simplified to

$$(p \wedge q \to p \vee q) \; .$$

The predicate logic introduces the quantifiers "for each" and "for some".

Notation	Reading	Conventional Notation
$\bigwedge x \underline{\mathrm{u}}x$	For each x, $\underline{\mathrm{u}}x$ is true	$(\forall x)\underline{\mathrm{u}}x$
$\bigvee x \underline{\mathrm{u}}x$	For some x, $\underline{\mathrm{u}}x$ is true	$(\exists x)\underline{\mathrm{u}}x$

In these expressions, ‘$\underline{\mathrm{u}}x$’ is to be thought of as a predicate about the object x. For example

$$(x + 1 = 3 \to x = 2)$$

Most of the theorems in these notes take the form $(p \to q)$. That is, there is a group of one or more hypotheses, followed by one or more conclusions. The logical symbols determine the sentence structure of nearly every theorem. There are exceptions such as Theorem R.14.3 of MI

$$(1 \in \mathrm{rp})$$

in which no logical symbol occurs. The first step the reader must take in learning to read the formal mathematics is to understand the logical structure of statements.

Morse's Set Theory

Morse's set theory includes proper classes and in particular a universe of sets, U, appearing in numerous theorems. A proper class cannot be an element of a class. A set is a class that is a member of some (other) class. In his book (but not in these notes), Morse uses the terms "set" and "point" to refer to classes and sets, respectively.

Morse's set theory uses both conventional and unconventional notation. Conventional notation includes:

Notation	Meaning
U	the universe
$(x \in y)$	x is a member of y
$(x \subset y)$	x is a subset of y
$(x \cup y)$	x union y
$(x \cap y)$	x intersect y
$\sim x$	the complement of x
$(x \backsim y)$	the set difference of x and y
(x, y)	the ordered pair of x and y

Note that U is both logical truth and the universal class. Similarly, $\sim x$ is both the logical negation of x and the class complement of x. For that matter, $(x \cap y)$ is equal to $(x \wedge y)$ and $(x \cup y)$ is equal to $(x \vee y)$. This suggests that $(x \to y)$ may be equal to $(\sim x \vee y)$, which it is.

The subset notation is sometimes used by other authors to mean proper subset, whereas Morse uses it in the sense of allowing equality. To indicate that x is a proper subset of y, Morse uses the notation $(x \subset\!\cdot\; y)$ This notation appears very infrequently.

As in the logic, the precedence of operators governs the omission of many parentheses. For example, the fully parenthesized statement

$$((x \in (A \cap B)) \to (x \in (A \cup B)))$$

may be simplified to

$$(x \in A \cap B \to x \in A \cup B)\ .$$

Another convential notation is illustrated by

$$(x \leq y \leq z = u \in A \subset B)\ .$$

This is a shorthand for

$$(x \leq y \wedge y \leq z \wedge z = u \wedge u \in A \wedge A \subset B)$$

which itself has been shortened by omission of parentheses. Another example is

$$(A \cap B \cap C)$$

which is shorthand for

$$((A \cap B) \cap C)\ .$$

Unconvential notations include the following.

Notation	Reading	Conventional Notation
0	the empty set	$\emptyset$
$\operatorname{sng} x$	singleton x	$\{x\}$
$\mathsf{E}\, x\underline{u}x$	the class of all elements x such that $\underline{u}x$ is true	$\{x : \underline{u}x\}$
$\bigwedge x\underline{u}x$	the intersection over x of $\underline{u}x$	$\bigcap x\underline{u}x$
$\bigvee x\underline{u}x$	the union over x of $\underline{u}x$	$\bigcup x\underline{u}x$
$\triangledown A$	the union of the members of A	$\bigcup A$
$\sqcap A$	the intersection of the members of A	$\bigcap A$
$\lambda\, x\underline{u}x$	the function taking each x to $\underline{u}x$	N/A
$\centerdot\, fx$	the functional value of f at x	$f(x)$

As is the case with notations discussed above, '0', '$\bigwedge x\underline{u}x$', and '$\bigvee x\underline{u}x$' have dual interpretations in logic and set theory. In any given instance it will be fairly evident which interpretation is intended.

The classifier symbol 'E' is a stylized E and is derived from the French word "ensemble", which is the French technical word for set. Polish logicians first began using this notation. The 'λ' symbol is a stylized lambda, called "lonzo" and pays homage to Alonzo Church, who invented the lamda notation for function definition that is widely used in logic and computer science. Morse's choice of classifier notation and "lonzo" notation helps to unify the notational treatment of bound variable forms. In particular it is often convenient to consider a bound variable form in which the bound variable is restriced to the members of a given class A. Examples are

Notation	Reading	Alternate Reading
$\bigwedge x \in A\underline{u}x$	for each x in A, $\underline{u}x$ is true	the intersection as x runs over A of $\underline{u}x$
$\bigvee x \in A\underline{u}x$	for some x in A, $\underline{u}x$ is true	the union as x runs over A of $\underline{u}x$
$\mathsf{E}\, x \in A\underline{u}x$	the set of x in A such that $\underline{u}x$ is true	
$\lambda\, x \in A\underline{u}x$	the function taking each x in A to $\underline{u}x$	

Other variations on this theme include

$$\mathsf{E}\, x \subset A\underline{u}x$$
$$\bigwedge x \in A \cup B\underline{u}x\ .$$

In conventional mathematics notation, juxtaposing two things may have any one of several different meanings, from function application to group multiplication. Morse adopts the convention that writing two things next to each other with no intervening operator is to be understood as meaning the intersection of the two things. Usage to mean intersection can be traced back to early formal logic in Boole. This usage occurs frequently in these notes.

It is common in mathematics that a defined term is meaningful only under certain conditions or in certain contexts. For example, an operation on an object may be meaningful only if the object is not a proper class. Thus, $\operatorname{sng} x$ can be the set with x as its sole member, only if x itself is a set. Similarly, the functional value of f at x, $\centerdot\, fx$, can fulfill the role only if x is a set. In such cases, it must be decided what value to assign to the term when the required condition is not met. In the case of $\operatorname{sng} x$, the empty set is the result if x is not a set. In the case of $\centerdot\, fx$, the result is the universe, U.

Word-style notation

Many of Morse's symbols are constructed as alphabetic combinations. It is easier to devise such combinations than to find a single character for each new concept. Furthermore, such combinations aid in remembering what concept is referred to. Examples of such symbols are:

Notation	Meaning
$\mathrm{sb}\, A$	the class of all subsets of A
$\mathrm{sp}\, A$	the class of all supersets of A
$\mathrm{sng}\, x$	singleton x; the set with sole member x
$\mathrm{singleton\,is}\, A$	A is a singleton; for some x, $(A = \mathrm{sng}\, x)$
$\mathrm{dmn}\, R$	the domain of R
$\mathrm{rng}\, R$	the range of R
$\mathrm{inv}\, R$	the inverse of R
$\mathrm{On}\, A$	the class of all functions with domain A

Rules of Syntax

According to Morse, a formal language is built on basic forms and rules of substitution. Each basic form is a linear array of symbols. For Morse's language, there are three types of symbols (1) variables, (2) constants, and (3) schemators (also known as schematic variables or second-order variables). The schematic variables are used in propositions of predicate logic as well as in many bound variable forms in mathematics. Most studies of formal mathematics limit the language to a first-order language containing no second-order variables. In Morse's language, only first-order variables are subject to quantification. The following are a few examples of basic forms with a categorization of the symbols of the form.

Basic Form	Variables	Constants	Schemators
$(p \to q)$	'p', 'q'	'(', '$\to$', ')'	
$(x \in A)$	'x', 'A'	'(', '$\in$', ')'	
$\bigwedge x \underline{\mathrm{u}} x$	'x'	'$\bigwedge$'	'$\underline{\mathrm{u}}$'
$\sum x \in A \underline{\mathrm{u}} x$	'x', 'A'	'$\sum$', '$\in$'	'$\underline{\mathrm{u}}$'
$\bigwedge x \bigvee y \underline{\mathrm{u}}' x y$	'x', 'y'	'$\bigwedge$'	'$\underline{\mathrm{u}}'$'
$\boldsymbol{.}\, f x$	'f', 'x'	'$\boldsymbol{.}$'	

All the schemators are symbols derived by priming from '$\underline{\mathrm{u}}$', '$\underline{\mathrm{v}}$', and '$\underline{\mathrm{w}}$'. Each schemator has an "arity" that is indicated by the number of primes attached to the schemator. A unary schemator has no primes, a binary schemator has one prime, a ternary schemator has two primes, etc. A *schematic expression* is a schemator followed by the number of distinct variables according to its arity. In basic forms, a schemator always appear in a schematic expression.

All of Morse's notations were devised to comply with his rules of syntax. The Polish logician Lesniewski did pioneering work in this area. Basic forms arise in one of two ways: either the basic form is a primitive undefined form or it is the left side of a definition. To ensure the readability of the language, definitions must follow rules of construction. Each definition in Morse's language is of the form '$(A \equiv B)$' where 'A' is replaced by a new basic form, the *definiendum*, to be defined and 'B' provides the definition of the new form, the *definiens*. There are rules for the construction of the definiendum and the definiens. These rules are intended to guarantee that no formula (well-formed expression) begins with a different formula. Thus in reading left to right, we will always know when we have come to the end of a formula.

Because a variable, such as x, by itself, is a well-formed expression, it follows that if a formula begins with a variable, then the variable is the formula. Because the letter f is a variable , it follows that the conventional function value notation '$f(x)$' cannot be a definiendum. Consequently, Morse adopted a non-conventional notation, '$\boldsymbol{.}\, f x$' for this purpose.

In relation to reading expressions from left to right, it should be noted that all of Morse's notations are linear. There are no vertical fractions, no variable occurs as a sub-script or super-script, etc. The restriction to linear notation complicates the effort to reproduce many conventional mathematical expressions such as summations and integrals. In Morse, the limits or range of summation or integration come to the right of the corresponding summation or integral symbol as shown below.

Notation	Reading	Conventional Notation
$\sum x \in A \underline{\mathrm{u}} x$	the sum over x in A of $\underline{\mathrm{u}}x$	$\sum_{x \in A} \underline{\mathrm{u}}x$
$\sup x \in A \underline{\mathrm{u}} x$	the supremum over x in A of $\underline{\mathrm{u}}x$	$\sup_{x \in A} \underline{\mathrm{u}}x$

Most of the basic forms that Morse uses fall into three categories, which we will refer to as (1) single constant, (2) infix operator, and (3) bound variable. The following table shows examples of these three.

Single Constant	Infix Operator	Bound Variable
sb A	$(p \to q)$	$\bigwedge x \underline{\mathrm{u}} x$
U	$(x \in A)$	⅄ $x \underline{\mathrm{u}} x$
Nb ra	$(x + y)$	$\mathsf{E}\, x \underline{\mathrm{u}} x$
sb A	(x, y)	$\sum x \underline{\mathrm{u}} x$

Part of Morse's language theory includes a thorough treatment of basic forms of the infix operator type. Such operators are assigned precedence levels to guide in the removal of parentheses. For the most part, these rules agree with the conventions of informal mathematical writing. The table below shows the precidence level of each of the binary operators that appear in these notes.

Precedence	Operators
2	'$\to$'
4	'$\leftrightarrow$'
5	'$\wedge$', '$\vee$'
6	'$\in$', '$\ni$', '$\subset$', '$\supset$', '$=$', '$\neq$', '$<$', '$>$', '$\leq$', '$\geq$', 'metrizes',
8	'$,$'
9	'$+$'
11	'$-$ '
13	'$/$'
15	'$\cap$', '$\cup$', '$\cdot$', '$\bullet$', '$\underline{\cap}$'
17	'$\backsim$'

The application of these rules to the comma operator in the ordered pair is not conventional. Thus, for example, in Morse's language, one may write

$$(x, y \in A)$$

as a shorthand for

$$((x, y) \in A) \ .$$

Putting the comma to further use, Morse uses the notation

$$(x, y, \in A)$$

as a shorthand for

$$((x, y), \in A)$$

which in turn is defined to mean

$$(x \in A \wedge y \in A) \ .$$

There are also rules established for variations of the bound variable forms, such as '$\sum x \in A\underline{\mathrm{u}}x$' and '$\sum x \in A \cup B\underline{\mathrm{u}}x$'. Combining this type of expression with the comma convention above, could result in an expression such as

$$\bigwedge x, y, \in A\underline{\mathrm{u}}'xy$$

which means the same as

$$\bigwedge x \in A \bigwedge y \in A\underline{\mathrm{u}}'xy\ .$$

Number Systems

As shown by Dedekind, the number systems may be built up using sets. These notes contain the results of such a construction, without exhibiting the actual steps. MI has the following number systems.

Notation	Meaning
ω	the natural numbers beginning with 0
ω'	the integers
rn	the rational numbers
rf	the finite real numbers
rl	the extended real numbers including $-\infty$ and ∞
rp	the positive extended real numbers
rfp	the positive finite real numbers

SI has all of the above along with the following.

Notation	Meaning
kf	the finite complex numbers
dinfin	the directed infinites; one for each direction in the complex plane
$\infty\!\!\!\!\!\mid\;$	the complex infinity $= 1/0$
infin	the set of all infinite numbers $=$ dinfin $\cup$ sng $\infty\!\!\!\!\!\mid\;$
spl	the summation plane $=$ kf $\cup$ sng $-\infty \cup$ sng ∞
cp	the complex plane $=$ kf $\cup$ sng $\infty\!\!\!\!\!\mid\;$
kt	the complex extension $=$ kf $\cup$ infin

In SI, the summation plane, spl, is the primary number system of interest.

The natural numbers are constructed, following von Neuman, beginning with 0 (the empty set) and following with

(1 = scsr 0 = 0 ∪ sng 0) where scsr is read "successor".

(2 = scsr 1)

(3 = scsr 2) etc.

The result of this construction is that each natural number is equal to the set of lower natural numbers.

One notable aspect of Morse's number systems, is that the number systems developed earlier have been embedded into the final results. While it is true (see SI 1.11.0 - 1.11.2) and not unexepected that

$$(\omega \subset \omega' \subset \mathrm{rn} \subset \mathrm{rf} \subset \mathrm{kf} \subset \mathrm{spl} \subset \mathrm{kt})$$
$$(\mathrm{rf} \subset \mathrm{rl} \subset \mathrm{spl})$$
$$(\mathrm{kf} \subset \mathrm{cp} \subset \mathrm{kt})$$

it might not be expected that the natural numbers retain their elementary set structure. Thus the complex number 0 and the real number 0 are both equal to the natural number 0 which in turn in equal to the empty set (and falsity).

It is important to understand the arithmetic of the number systems. Note the completely general theorems of commutativity and associativity where no assumption is made that x, y, and z are finite numbers or numbers of any kind (see MI R.7.0 - R.7.3 and SI 1.6.6 - 1.6.8).

$$(x + y = y + x)$$
$$(x \cdot y = y \cdot x)$$
$$(x + (y + z) = x + y + z = (x + y) + z)$$
$$(x \cdot (y \cdot z) = x \cdot y \cdot z = (x \cdot y) \cdot z)$$

There are many cases where $(x + y)$ and $(x \cdot y)$ do not have useful meanings. In such cases, the value is the universe, U. If x and y are infinite numbers, their sum is a number only if they are the same directed infinity, while their product is always a number. Any finite complex number added to an infinite number equals the infinite number. Any non-zero finite complex number times an infinite number yields an infinite number.

The desire to have unconditional associativity is one of the reasons behind directed infinities. If plus and minus infinity are included in the arithmetic and if

$$(-\infty = -1 \cdot \infty) ,$$

then by associativity it follows that

$$(-\infty = -1 \cdot \infty = (\mathrm{i} \cdot \mathrm{i}) \cdot \infty = \mathrm{i} \cdot (\mathrm{i} \cdot \infty)) .$$

Thus $(\mathrm{i} \cdot \infty)$ must have some meaning.

Similarly, the following example shows that including infinity in arithmetic prevents an unconditional theorem of distributivity.

$$((0 + 1) \cdot \infty = 1 \cdot \infty = \infty \neq \mathrm{U} = \mathrm{U} + \infty = 0 \cdot \infty + 1 \cdot \infty)$$

Morse uses the notation $^{\bullet}xn$ to indicate raising the number x to the integer power n. In particular, any number x raised to the 0 power is equal to 1. This includes zero to the zero power ($^{\bullet}00$), which is often viewed as indeterminate. One of the advantages of Morse's answer is that it makes power series, such as that for the exponential, work. In order for exp(0), i.e. e to the 0 power, to equal 1, the first term in the series must equal 1, which in turn requires that 0 to the 0 power is equal to 1.

Morse has extended the use of $(x + y)$ and $(x \cdot y)$ to work when x and y are number-valued functions, or when one of x or y is a number-valued function and the other is a number (see MI R.11 and SI 1.10). In fact, the Ph. D. thesis of one of his students, Robert Arnold, was devoted to further extensions of plus and times.

There are bound variable and non-bound variable forms for infima and suprema.

Notation	Meaning
$\inf x \in A \underline{\mathrm{u}} x$	the infinum of $\underline{\mathrm{u}} x$ as x runs over A
$\sup x \in A \underline{\mathrm{u}} x$	the supremum of $\underline{\mathrm{u}} x$ as x runs over A
$\mathrm{Inf}\, A$	the infimum of the set of numbers A
$\mathrm{Sup}\, A$	the supremum of the set of numbers A

Morse defined the absolute value, $|x|$, in a somewhat unusual way, so that the absolute value is always a number from 0 to ∞. In case x is not a number, the absolute value of x is equal to ∞. This facilitates some nice theorems about sums and integrals of absolute values that require very little, if any, hypotheses. Examples are the following theorems from SI.

1.16.0 $(0 \leq |x| \leq \infty)$
1.16.3 $(|-x| = |x|)$
1.16.4 $(|x + y| \leq |x| + |y|)$
1.16.5 $(||x| - |y|| \leq |x - y|)$

Convergence via Runs

There have been several attempts to unify the concept of topological convergence. For example, Bourbaki *General Topology, Chapters 1-4* (Springer Verlag, 1989) put forth the theory of filters (p. 57) and Kelley, in his book *General Topology* (Van Nostrand, 1955), used the theory of nets(p. 62) as first developed by Moore and Smith. Kelley stated "We are interested in developing a theory which will apply to convergence of sequences, of double sequences, to summation of sequences, to differentiation and integration.". Morse, along with Hewitt Kenyon, developed the concept of a *run* as a generalization of a direction (see *Runs*, H. Kenyon and A. P. Morse, Pacific Journal of Mathematics, 1958).

A direction is a non-empty transitive relation, R, so that for any two members x and y of the domain of R, there is a z in the domain of R such that

$$(x, z \in R \wedge y, z \in R) \ .$$

Because Morse saw that some useful applications of runs would require that the "elements" x, y, and z, may be proper classes (and therefore incapable of belonging to a set), he replaced the notion of ordering with a comparison of "vertical sections" of a relation. If R is a relation (set of ordered pairs), then the "vertical section of R at x" is the set of all y such that the ordered pair (x, y) is a member of R.

$$(\mathrm{vs}\, Rx \equiv \sqsubseteq y(x, y \in R))$$

Then the defining characteristic of a run is that for any two members x and y of the domain of R, there is a z in the domain of R such that

$$(\mathrm{vs}\, Rz \subset \mathrm{vs}\, Rx \cap \mathrm{vs}\, Ry) \ .$$

As an example, Morse defines the run 'sumrun' used for unordered summation as

$$(\mathrm{sumrun} \equiv \sqsubseteq \alpha, \beta(\alpha \subset \beta \in \mathrm{fnt})) \ .$$

That is, sumrun is the set of all ordered pairs of finite sets such that the first coordinate is a subset of the second coordinate. The vertical sections of this relation are proper classes.

What is most notable about runs, however, is the fact that the domain and range of the run need not have any members in common. Clearly this cannot be true for a direction, because the z that is found for x and y is in the domain and in the range. However, for the purposes of integration, Morse defines the run, called 'mode φ', in which members of the domain are partitions of a given set while members of the range are selector functions defined on partitions. This run is discussed in the section below on integration. In short, runs retain all of the needed characteristics of directions, but are considerably more flexible regarding proper classes and the separation of domain and range.

Runs can be used to define limits for all of the purposes cited above by Kelley. The most important limit concept used in these notes is the limit in the summation plane denoted by

$$\mathrm{lm}\, xR\underline{\mathrm{u}}x \ .$$

This is the limit in the summation plane of the values $\underline{\mathrm{u}}x$ as x runs along R. In this context, R is a run to be specified. For x to run along R means that x is in the range of R. The limit has the property that it is equal either to a number in the summation plane or to the universe. That is

$$(L = \mathrm{lm}\, xR\underline{\mathrm{u}}x \rightarrow L \in \mathrm{spl} \vee L = \mathrm{U}) \ .$$

It may be helpful to state an ϵ - δ type condition for the value of this limit.

$$(\mathrm{run\,is}\, R \wedge L \in \mathrm{spl} \rightarrow L = \mathrm{lm}\, xR\underline{\mathrm{u}}x \leftrightarrow \bigwedge \epsilon \in \mathrm{rfp} \bigvee \delta \in \mathrm{dmn}\, R \bigwedge x \in \mathrm{vs}\, R\delta(\underline{\mathrm{u}}x \in \mathrm{Nb}\, \epsilon L))$$

Summation

Morse's theory of summation is very carefully crafted and the two courses give different perspectives on that development. In MI, summation is developed from a postulational basis and deals only with real numbers while in SI, the full development of summation is given based on complex numbers. The following notations are used in different ways in the two courses in connection with summation.

Notation	Meaning	MI	SI
$\mathrm{Nb}\, rp$	r neighborhood of p	p must be real	p may be complex
$\mathrm{lm}\, xR\underline{\mathrm{u}}x$	R limit of $\underline{\mathrm{u}}x$	$\underline{\mathrm{u}}x$ must be real	$\underline{\mathrm{u}}x$ may be complex
$\mathrm{lin}\, n\underline{\mathrm{u}}n$	sequential limit of $\underline{\mathrm{u}}n$	$\underline{\mathrm{u}}n$ must be real	$\underline{\mathrm{u}}n$ may be complex
$\sum x \in A\underline{\mathrm{u}}x$	sum over x in A of $\underline{\mathrm{u}}x$	$\underline{\mathrm{u}}x$ must be real	$\underline{\mathrm{u}}x$ may be complex
$\underset{\sim}{\sum} n\underline{\mathrm{u}}n$	symmetric sequential sum	$\underline{\mathrm{u}}n$ must be real	$\underline{\mathrm{u}}n$ may be complex

Morse's approach is somewhat unusual in that he develops unordered summation as the fundamental form of summation. The development in SI is done as follows.

Topic	Notation	Begin	End
Finite unordered summation	$\mathrm{ad}\, x \in \alpha\underline{\mathrm{u}}x$	1.58	1.64
Infinite unordered summation of reals	$\mathrm{Ad}\, x\underline{\mathrm{u}}x$	1.65	1.72.1
General unordered infinite summation	$\sum x\underline{\mathrm{u}}x$	1.72.2	1.86
Infinite unordered summation over a class	$\sum x \in A\underline{\mathrm{u}}x$	1.87	1.134
Symmetric ordered summation	$\underset{\sim}{\sum} n\underline{\mathrm{u}}n$	1.135	1.139

It is only when we get to Symmetric Summation, after 77 defintions, lemmas, and theorems, that we encounter the special case of ordered summation. This is the only form of summation treated in many text books.

In the development of summation and integration, Morse uses a special form of multiplication

$$(x \bullet y)$$

which equals $(x \cdot y)$ if neither x nor y is 0. If either x or y is 0, then $(x \bullet y)$ is equal to 0. This implies, for example, that $(\mathrm{U} \bullet 0 = 0)$. The motivation behind this device is to ensure some results with minimal hypotheses and it facilitates defining summation and integration over a given set as special cases of general summation and integration.

As indicated in the table above, the study of summation begins with finite summation, that is, the sum of a finite number of things. At this stage, it is not required that the things summed belong to the summation plane for the sum to exist (i.e., not equal to U).

Summation of real numbers (including infinite real numbers) is defined as a limit of finite sums. In this case, the positive numbers and negative numbers are summed separately and the final answer is the combination. For this purpose, Morse sets up the concepts of $\mathrm{ps}\, x$ and $\mathrm{ng}\, x$ which are the nonnegative part and the nonpositive part of x, respectively. Each is defined as a nonnegative number and so that if x is not a real number, then the answer is ∞. Thus, for example,

x	$\mathrm{ps}\, x$	$\mathrm{ng}\, x$
0	0	0
1	1	0
-1	0	1
∞	∞	0
$-\infty$	0	∞
U	∞	∞

As mentioned above, a special run, called sumrun is set up for the purpose of defining real summation. This run is the class of ordered pairs of finite sets (α, β) for which $(\alpha \subset \beta)$. We are farther along in sumrun when we are considering larger and larger finite sets. For sumrun we can restate the ϵ - δ condition for the value of the limit as follows.

$$(R = \text{sumrun} \wedge L \in \text{spl} \rightarrow L = \text{lm}\, \alpha R \underline{\text{u}}\alpha \leftrightarrow \bigwedge \epsilon \in \text{rfp} \bigvee \delta \in \text{fnt} \bigwedge \alpha \in \text{fnt}(\delta \subset \alpha \rightarrow \underline{\text{u}}\alpha \in \text{Nb}\, \epsilon L))$$

Using these concepts, summation of reals is defined as follows.

$$(\text{Ad}\, x \underline{\text{u}} x \equiv (\text{lm}\, \alpha\, \text{sumrun}\, \text{ad}\, x \in \alpha\, \text{ps}\, \underline{\text{u}} x \text{ --- } \text{lm}\, \alpha\, \text{sumrun}\, \text{ad}\, x \in \alpha\, \text{ng}\, \underline{\text{u}} x))\ .$$

It is worthwhile noting that limits are not needed at this stage. Each of the limits in the above definition could be replaced by a supremum over finite sets, since in each case only nonnegative numbers are being summed. Morse acknowledges this in theorem 1.70 which follows from 1.51.0.

In order to define summation over numbers in the summation plane (finite complex, ∞, and $-\infty$), Morse uses the concepts of the real and imgainary parts of a number. If x is a finite complex number (i.e., $(x \in \text{kf})$), then there are unique finite real numbers a and b so that

$$(x = a + \text{i} \cdot b)\ .$$

Morse denotes the a and b by $\text{prt}'\, x$ and $\text{prt}''\, x$, respectively. For the infinite reals he sets

$$(\text{prt}'\, \infty = \infty \wedge \text{prt}''\, \infty = 0 \wedge \text{prt}' - \infty = -\infty \wedge \text{prt}'' - \infty = 0)\ .$$

With these concepts, general unordered summation is defined as follows.

$$(\textstyle\sum x \underline{\text{u}} x \equiv (\text{Ad}\, x\, \text{prt}''\, \underline{\text{u}} x \in \text{rf} \rightarrow \text{Ad}\, x\, \text{prt}'\, \underline{\text{u}} x + \text{i} \cdot \text{Ad}\, x\, \text{prt}''\, \underline{\text{u}} x))$$

An interesting theorem is the following.

$$(\textstyle\sum x \underline{\text{u}} x \in \text{spl} \rightarrow \sum x \underline{\text{u}} x = \text{lm}\, \alpha\, \text{sumrun}\, \text{ad}\, x \in \alpha \underline{\text{u}} x)$$

This theorems suggests the possibility that it was not necessary to use the intermediate concept of real summation in order to define the final summation. That is, might it be the case that the following is a theorem?

$$(\textstyle\sum x \underline{\text{u}} x = \text{lm}\, \alpha\, \text{sumrun}\, \text{ad}\, x \in \alpha \underline{\text{u}} x)$$

The following example, not given in the notes, shows this to be false.

$$(\bigwedge x((x = 0 \rightarrow \underline{\text{u}} x = \infty) \wedge (x \neq 0 \rightarrow \underline{\text{u}} x = -1) \rightarrow \textstyle\sum x \underline{\text{u}} x = \text{U} \wedge \text{lm}\, \alpha\, \text{sumrun}\, \text{ad}\, x \in \alpha \underline{\text{u}} x = \infty)$$

This is because as soon as 0 is included in the finite sets, the finite sum will always be ∞, hence the limit of the finite sums is ∞. On the other hand, when considering the positive and negative parts, each sum is infinite and the combination is equal to U. This shows that even real summation is not a simple limit of finite sums. That is, the following is not a theorem.

$$(\bigwedge x(\underline{\text{u}} x \in \text{rl}) \wedge \text{lm}\, \alpha\, \text{sumrun}\, \text{ad}\, x \in \alpha \underline{\text{u}} x = S \in \text{rl} \rightarrow S = \text{Ad}\, x \underline{\text{u}} x).$$

What can be said is the following (which is close to 1.133).

$$(\text{lm}\, \alpha\, \text{sumrun}\, \text{ad}\, x \in \alpha \underline{\text{u}} x = S \in \text{kf} \rightarrow \textstyle\sum x \underline{\text{u}} x = S)$$

Whenever the sum exists, it is the limit of finite sums. Therefore, it may be helpful to state an ϵ - δ type condition for the value of summation.

$$(S \in \text{spl} \rightarrow S = \textstyle\sum x \underline{\text{u}} x \leftrightarrow \bigwedge \epsilon \in \text{rfp} \bigvee \delta \in \text{fnt} \bigwedge \alpha \in \text{fnt}(\delta \subset \alpha \rightarrow \text{ad}\, x \in \alpha \underline{\text{u}} x \in \text{Nb}\, r S))$$

For restricted summation, the condition is the following.

$$(S \in \text{spl} \rightarrow S = \textstyle\sum x \in A \underline{\text{u}} x \leftrightarrow \bigwedge \epsilon \in \text{rfp} \bigvee \delta \in \text{fnt} \cap \text{sb}\, A \bigwedge \alpha \in \text{fnt} \cap \text{sb}\, A(\delta \subset \alpha \rightarrow \text{ad}\, x \in \alpha \underline{\text{u}} x \in \text{Nb}\, r S))$$

Measure Theory

As mentioned earlier, the development of measure theory appears only in the MI notes. A measure on a set S is a non-negative real-valued function, φ, on the subsets of S. Most authors define a measure as being defined on a σ-field of subsets of S. Morse defines the set of φ-measurable subsets of S as those subsets that satisfy the Caratheodory criteria

$$(\text{mbl}\, \varphi \equiv \text{E}\, A \subset S \bigwedge T \subset S(\centerdot\, \varphi T = \centerdot\, \varphi(TA) + \centerdot\, \varphi(T \sim A)))$$

After developing the general results of measure theory, Morse turns to studying measures on metric spaces. Again, Morse's definition is slightly at odds with the usual approach. Most authors define a metric space so that if the distance between x and y is 0, then x equals y. Morse's relaxes this rule, so that his definition of metric space is what others refer to as a pseudo-metric space. While he defines the notion of simple metric where the condition is introduced, this notion does not appear in any of the theorems.

It is worth while to notice the technique employed in Definitions 2.50.0 and 2.81.0.

2.50.0 $(\mathrm{dmtr}\,\rho n \equiv \text{E}\,\beta(\rho \in \mathrm{metric} \wedge \mathrm{diam}\,\rho\beta \leq \underline{2}\,n))$

2,81.0 $(\mathrm{bore}\,\rho \equiv \text{E}\,\beta(\rho \in \mathrm{metric} \wedge \bigwedge \varphi \in \mathrm{Md}\,\rho(\beta \in \mathrm{mbl}\,\varphi)))$

Each of these has a definiens that begins "$\text{E}\,\beta(\rho \in \mathrm{metric} \wedge$..." . The purpose of this construction is to ensure that the assertion $(\beta \in \mathrm{dmtr}\,\rho n)$ or $(\beta \in \mathrm{bore}\,\rho)$ will imply that $(\rho \in \mathrm{metric})$, so that this condition does not have to be stated separately.

Integration

The development of integration in SI is quite abstract and we will attempt to describe this development, along with describing the more concrete case given in MI. The concept under discussion is the integration of a numerical-valued function, f, over a space, S, with respect to a measure-like function, φ. The function φ is defined on subsets of the space S whereas the function f is defined on the points of S. Thus we may consider that the space S is determined either by the function f, by $(S = \mathrm{dmn}\,f)$, or by the function φ, by $(S = \nabla\,\mathrm{dmn}\,\varphi)$. Morse has chosen the latter, to allow more freedom in the domain of f.

The basic idea of the integral is to define Riemann-type sums as approximations to the integral as follows. The process begins with a partition, D, of the space S that does not contain the empty set. Next comes a "selector" function, ξ, defined on D $(D = \mathrm{dmn}\,\xi)$, so that

$$\bigwedge \beta \in D(\centerdot\,\xi\beta \in \beta)\ .$$

The Riemann sum is now defined as

$$(\mathrm{rsum}\,f\xi\varphi \equiv \sum \beta \in \mathrm{dmn}\,\xi(\centerdot\,f\,\centerdot\,\xi\beta \bullet \centerdot\,\varphi\beta))\ .$$

Thus, for each member, β of the partition, the selector function chooses an element, x, in β, namely $(x = \centerdot\,\xi\beta)$. The function f is evaluated at the point x and then multiplied by the "measure" of the set β, giving $(\centerdot\,fx \bullet \centerdot\,\varphi\beta)$. The multiplication here gives 0 whenever either of the two mulitiplicands equals 0. The resulting products are summed over all the members of the partition to obtain the Riemann sum.

The integral is defined as a limit of the Riemann sums, according to some, as yet unspedified, method of integration. The method of integration is represented by a run M, and a preliminary integral is defined as

$$(\textstyle\int\# \, Mf\varphi\} \equiv \mathrm{lm}\,\xi M\,\mathrm{rsum}\,f\xi\varphi)\ .$$

In MI, φ must be a measure for the integral to exist. However, in the initial stage of SI, no restriction whatever is placed on f, ξ, or φ in the definition of the Riemann sum and no restriction is placed on M for the definition of the integral. That is, f is not assumed to be a numerically valued function, ξ is not assumed to be a selector function, and φ is not assumed to be a measure-type function or a function of any kind. Even without such restrictions, an ϵ - δ type condition can be stated for the existence and value of this very general preliminary integral.

$$(\mathrm{run\,is}\,M \wedge J \in \mathrm{spl} \rightarrow J = \textstyle\int\# \, Mf\varphi\} \leftrightarrow \bigwedge \epsilon \in \mathrm{rfp} \bigvee \delta \in \mathrm{dmn}\,M \bigwedge \xi \in \mathrm{vs}\,M\delta(\mathrm{rsum}\,f\xi\varphi \in \mathrm{Nb}\,\epsilon J))$$

Before considering a specific method of integration, Morse refines the preliminary integral with regard to the infinite cases. He requires that if the integral is infinite, then the function being integrated must be the limit of a sequence of functions, each of which has a finite integral. A function satisfying this condition is said to be "neared". The fact is expressed using the notation

$$\mathrm{neared}\#\,M\varphi f\ .$$

Using this concept a second preliminary integral is defined as follows:

$$(\textstyle\int\# \, Mf\varphi] \equiv (\mathrm{neared}\,M\varphi f \rightarrow \textstyle\int\# \, Mf\varphi\}))\ .$$

The "implies" sign ('$\rightarrow$') in this definition has the effect that if f is neared, then the result is the preliminary intergral; whereas if f is not neared, then the result is the universe, U. Thus, f must be neared for the integral to exist (be a number in the summation plane).

The new ϵ - δ condition is not much changed.

$$(\text{neared}\, M\varphi f \wedge J \in \text{spl} \rightarrow$$
$$J = \int\#\, Mf\varphi] \leftrightarrow \bigwedge \epsilon \in \text{rfp} \bigvee \delta \in \text{dmn}\, M \bigwedge \xi \in \text{vs}\, M\delta(\text{rsum}\, f\xi\varphi \in \text{Nb}\,\epsilon J))$$

A bound variable notation for the integral is defined at this stage as follows

$$(\int\#\, M\underline{\text{u}}x\varphi\,\text{d}\,x \equiv \int\#\, M \curlywedge x\underline{\text{u}}x\varphi]) \ .$$

The only method of integration discussed in either of the two courses is integration by refinement of the partition. If D and D' are partitions of S, then D' is a refinement of D, denoted by '$(D' \subset\subset D)$', if each member of D' is a subset of some member of D. The integral is defined as the limit of the Riemann sums, as the partitions become more and more refined.

An example of an alternative method of integration can be found in nearly any calculus text book, where Riemann integration over a finite interval is defined. The interval is partitioned into a finite number of subintervals and the "mesh" of the partition is defined as the maximum length of the subintervals. The limit method in this case is to require that the mesh of the partition approaches zero. The general structure set up here easily accomodates such an alternative method.

In the context of integration by refinement, conditions on the arguments begin to appear. The only Riemann sums, rsum $f\xi\varphi$, that will be considered will require that ξ is a selector function and that φ is in the class of 'grator' functions. A function, φ, in grator is a function whose range is a subset of the summation plane and that has the value 0 at the empty set. Keep in mind that the space of integration, S, is determined by φ as $(S = \nabla\, \text{dmn}\, \varphi)$. Morse defines the term 'rlm φ' (realm φ) to refer to $\nabla\, \text{dmn}\, \varphi$. The domain of the selector function ξ is required to be a countable partition of S (i.e., a disjoint countable collection, D, of nonempty sets, such that $(\nabla D = S)$). Furthermore the only such partitions, D, that are considered are those on which φ is additive in the following sense. For each T in the domain of φ, it must be the case that

$$(\centerdot\, \varphi T = \sum \beta \in D\, \centerdot\, \varphi(T\beta)) \ .$$

This is a very strong condition for it implies that for each β in D, $(T\beta)$ is in dmn φ. (Recall that $(T\beta = T\cap\beta)$.) The set of partitions that satisfy these conditions is called 'scheme φ' and is defined as follows

$$(\text{scheme}\, \varphi \equiv \sqsubseteq D \in \text{partion rlm}\, \varphi(\varphi \in \text{grator} \wedge \bigwedge T \in \text{dmn}\, \varphi(\centerdot\, \varphi T = \sum \beta \in D\, \centerdot\, \varphi(T\beta)))) \ .$$

Here partion rlm φ is the set of all countable disjoint partitions of rlm φ, consisting of non-empty sets.

The conditions on scheme φ are not sufficient for Morse's purposes. He defines a special type of scheme called a grid as follows.

$$(\text{grid}\, \varphi \equiv \sqsubseteq D \in \text{scheme}\, \varphi \bigwedge G \in \text{scheme}\, \varphi(D \sqcap G \in \text{scheme}\, \varphi))$$

where $(D \sqcap G)$ is a common refinement of D and G formed by taking non-empty intersections of members of D with members of G. If φ is a measure, then the members of scheme φ are the countable partitions of S of φ-measureable sets and

$$(\text{grid}\, \varphi = \text{scheme}\, \varphi) \ .$$

In fact, theorem 7.36 shows that every finite scheme is a grid and theorem 7.37 shows that if φ is a non-negative function taking 0 to 0, then $(\text{grid}\, \varphi = \text{scheme}\, \varphi)$. Thus the two concepts coincide under much more general conditions than φ being a measure.

With the above conditions placed on the partitions and the grator φ, the ground is laid to spell out the refinement method of integration. The refinement run is defined, denoted by 'mode φ' as

$$(\text{mode}\, \varphi \equiv \sqsubseteq D, \xi(D \in \text{grid}\, \varphi \wedge \xi \in \text{selector} \wedge \text{dmn}\, \xi \in \text{grid}\, \varphi \wedge \text{dmn}\, \xi \subset\subset D)) \ .$$

Thus, if $(D, \xi \in \text{mode}\, \varphi)$ then the following statements are true.

$$(D \in \text{grid}\, \varphi \wedge D \in \text{scheme}\, \varphi \wedge \nabla D = \text{rlm}\, \varphi = \nabla\, \text{dmn}\, \varphi)$$
$$(\xi \in \text{selector} \wedge \text{dmn}\, \xi \in \text{grid}\, \varphi \wedge \text{dmn}\, \xi \subset\subset D)$$

Theorem 7.34 shows that if φ is in grator then mode φ is a run. Using mode φ, Morse defines a non-preliminary, but still very general, form of integral

$$(\int f\varphi\} \equiv \int\#\ \text{mode}\, \varphi f\varphi]) \ ,$$

along with a bound variable form

$$(\int \underline{\text{u}}x\varphi\,\text{d}\,x \equiv \int\#\ \text{mode}\, \varphi\underline{\text{u}}x\varphi\,\text{d}\,x) \ .$$

We conclude this discussion with an ϵ - δ characterization of the above bound variable form of the integral. In preparation for our characterization, we give the specialized version of "neared" as it applies to integration by refinement.

$$(\mathrm{neared}\, \varphi f \equiv \mathrm{neared}\#\, \mathrm{mode}\, \varphi\varphi f)$$

Now our characterization is

$$\begin{aligned}(f = \measuredangle x\underline{\mathrm{u}}x \wedge \varphi \in \mathrm{grator} \wedge M = \mathrm{mode}\, \varphi \wedge \mathrm{neared}\, \varphi f \wedge J \in \mathrm{spl} \rightarrow \\ J = \textstyle\int \underline{\mathrm{u}}x\varphi \,\mathrm{d}\, x \leftrightarrow \bigwedge \epsilon \in \mathrm{rfp} \bigvee \delta \in \mathrm{grid}\, \varphi \bigwedge \xi \in \mathrm{vs}\, M\delta(\mathrm{rsum}\, f\xi\varphi \in \mathrm{Nb}\, \epsilon J)\ .\end{aligned}$$

This can be made a little more specific by expanding on the meaning of 'mode φ' as follows.

$$\begin{aligned}&(f = \measuredangle x\underline{\mathrm{u}}x \wedge \varphi \in \mathrm{grator} \wedge \mathrm{neared}\, \varphi f \wedge J \in \mathrm{spl} \rightarrow \\ &\quad J = \textstyle\int \underline{\mathrm{u}}x\varphi \,\mathrm{d}\, x \leftrightarrow \\ &\quad \bigwedge \epsilon \in \mathrm{rfp} \bigvee \delta \in \mathrm{grid}\, \varphi \bigwedge \xi \in \mathrm{selector} \\ &\qquad\qquad (\mathrm{dmn}\, \xi \in \mathrm{grid}\, \varphi \wedge \mathrm{dmn}\, \xi \subset\subset \delta \rightarrow \mathrm{rsum}\, f\xi\varphi \in \mathrm{Nb}\, \epsilon J)\ .\end{aligned}$$

Completely Additive Functions

Each of the two courses has a section with this title. In MI, the study is restricted to what might be called signed measures, while in SI, complex valued measures are considered. The Hahn and Jordan decomposition theorems are the primary results of these sections. It is notable that Morse has defined completely additive functions in such a way that the sum of two of them is again a completely additive function. This is not usually the case (cf. Halmos, *Measure Theory*, Van Nostrand, 1950, pp. 117-118).

Some Special Notations

It was mentioned earlier that Morse's language does not distinguish between terms and formulas. There are a few definitions that make use of this unification and thus the reader might benefit from an explanation of some of these.

The following references are from SI.

1.9 $(\mathrm{noz}\, x \equiv (x = 0 \vee x))$. The definiens is $(x = 0 \vee x)$. Using the rules for omission of parentheses, this is equivalent to $((x = 0) \vee x)$. In this case, the "or" sign is acting as a set union. The definition could have been written using the set union sign in place of the "or" sign, but the additional parentheses would have been required in that case. This is because the "or" sign is more fundamental in the expression than a union sign would be. In particular "or" is more fundamental than '=' while union is less fundamental than '='. On the left is the statement '$(x = 0)$' which is either true or false and thus has either the value 0 or the value U. On the right is x. Thus, if $(x = 0)$, the result is $((\mathrm{U} \vee x) = \mathrm{U} \cup x = \mathrm{U})$. On the other hand, if $(x \neq 0)$, i.e. $\sim(x = 0)$, then $(x = 0)$ is equal to 0 and the result is $((0 \vee x) = 0 \cup x = x)$. Therefore $\mathrm{noz}\, x$ ("non-zero x") is x if x is not 0 and U (the universe) if X is 0.

1.12 $(\sqrt{x} \equiv (0 \leq x \rightarrow \mathrm{Sup} \sqsubseteq t \geq 0(^{\bullet}t2 \leq x)))$. Again, using the rules for omission of parentheses, the definiens is equivalent to $((0 \leq x) \rightarrow \mathrm{Sup} \sqsubseteq t \geq 0(^{\bullet}t2 \leq x))$. In this case, we can think of an implication $(p \rightarrow q)$ in terms of truth values. The implication is true whenever either p is false or q is true. Restated in logical terms, $(p \rightarrow q)$ is the same as $(\sim p \vee q)$, not p or q. In set theory terms, it is the same as $(\sim p \cup q)$, complement p union q. Thus we could rewrite the definiens as $(\sim(0 \leq x) \cup \mathrm{Sup} \sqsubseteq t \geq 0(^{\bullet}t2 \leq x))$. If $(0 \leq x)$, then $\sim(0 \leq x)$ is equal to 0 and the definiens is equal to $\mathrm{Sup} \sqsubseteq t \geq 0(^{\bullet}t2 \leq x)$. On the other hand, if $(0 \leq x)$ is false then, $\sim(0 \leq x)$ is equal to U and the definiens is equal to U. Thus the construction of the definiens assures that we will get the (non-negative) square root of x, if x is non-negative and we will get the universe otherwise.

1.14.1 $(\mathrm{prt}''\, x \equiv (x \backsim\in \mathrm{rl} \wedge \mathrm{The}\, b \in \mathrm{rf} \bigvee a \in \mathrm{rf}(x = a + \mathrm{i}\cdot b)))$. With parentheses back in, the definiens becomes $((x \backsim\in \mathrm{rl}) \wedge \mathrm{The}\, b \in \mathrm{rf} \bigvee a \in \mathrm{rf}(x = a + \mathrm{i}\cdot b))$. Here the "and" sign operates as intersection and has been used in place of intersection because of its level of precedence for parenthesis removal.

1.14.2 $(|x| \equiv (x \in \mathrm{kf} \wedge \surd({}^{\bullet}\,\mathrm{prt}'\, x2 + {}^{\bullet}\,\mathrm{prt}''\, x2) \vee x \backsim\in \mathrm{kf} \wedge \infty))$. To help analyze the definiens, let us use the letter A to represent the expression $\surd({}^{\bullet}\,\mathrm{prt}'\, x2 + {}^{\bullet}\,\mathrm{prt}''\, x2)$. Then the definiens becomes $(x \in \mathrm{kf} \wedge A \vee x \backsim\in \mathrm{kf} \wedge \infty)$. WIth parentheses back in, the definiens is $(((x \in \mathrm{kf}) \wedge A) \vee ((x \backsim\in \mathrm{kf}) \wedge \infty))$. Now the entire definiens can be seen as one big "or" value. Analyzing this like the previous example shows that if $(x \in \mathrm{kf})$ then the absolute value of x is A and if $(x \backsim\in \mathrm{kf})$ then the absolute value of x is ∞.

1.21 $(\mathrm{Nb}\, ra \equiv (a = -\infty \wedge \sqsubseteq x(\mathrm{prt}'\, x \le -1/r) \vee a \in \mathrm{kf} \wedge \sqsubseteq x(|x - a| \le r) \vee a = \infty \wedge \sqsubseteq x(\mathrm{prt}'\, x \ge 1/r)))$. To analyze the definiens, let us use the abbreviations A for $\sqsubseteq x(\mathrm{prt}'\, x \le -1/r)$, B for $\sqsubseteq x(|x — a| \le r)$, and C for $\sqsubseteq x(\mathrm{prt}'\, x \ge 1/r)$. Then the definiens, with parentheses added, becomes $((((a = -\infty) \wedge A) \vee ((a \in \mathrm{kf}) \wedge B)) \vee (a = \infty \wedge C))$. In this case the statement is a big "or" statement based on the second "or". With parentheses added, the "or"s can be replaced by unions and logical analysis shows that: if a is equal to negative infinity, then the result is A; if a is a finite complex number, then the result is B; if a is positive infinity, then the result is C; and in any other case, the result is 0. The definiendum can be read as "neighborhood ra" or the r neighborhood of a. A similar analysis applies to 1.22.0.

1.87 $((a \bullet b) \equiv (a \ne 0 \wedge b \ne 0 \wedge a\cdot b))$. With parentheses added, the definiens is $((a \ne 0) \wedge (b \ne 0) \wedge (a\cdot b))$. Thus as long as both $(a \ne 0)$ and $(b \ne 0)$ are true (and thus equal to the universe), then the result is $(a \cdot b)$. However, if either $(a = 0)$ or $(b = 0)$ is true, then the result is 0. This essentially means that multiplying anything by 0 gives 0, for this special form of multiplication.

1.92 $(\mathrm{Cr}\, xy \equiv (x \in y \wedge 1))$. The revised definiens is $((x \in y) \wedge 1)$. If $(x \in y)$ is true then the result is 1, whereas if $(x \in y)$ is false then the result is 0. This is the characterstic function for the set y applied to x. The result is 1 or 0 depending on whether x is an element of y or not.

A Few Additional Notations

There are a few other notations in SI which are either not explained or merit some comment. In the case of non-explained notations, it may be that Morse had some additional material for the course, such as the Background Notation for MI. On the other hand, the reproduction of these notes is based on a handwritten copy done by Bob Alps in 1971 and it is possible that unexplained notation was introduced during the copying process. For example, Morse's notes typically did not include the outer parentheses formally needed in most theorems. These were added in the copying process to be consistent with the standard of formality in Morse's book. Some notations from the book may have found their way into the notes in a similar fasion.

1.1.0 scsr x - "successor x" equal to $(x \cup \operatorname{sng} x)$. This is the successor function for natural numbers. Zero is the empty set, 0. One is the successor of zero,

$$(1 = \operatorname{scsr} 0 = 0 \cup \operatorname{sng} 0 = \operatorname{sng} 0) \ ,$$

etc. Each natural number is the set of all lower natural numbers.

1.1.0 $(x \in y)$ - membership relation, x is a member of the class y; x is an element of the class y.

1.1.5 $(x \cup y)$ - union, x union y, the class with elements from either x or y.

1.1.11 $(x \cap y)$ - intersection, x intersect y, the class with elements in both x and y.

1.2.2 $(x \subset y)$ - subset, x is a subset of y, each element of x is also an element of y.

1.8.0 $^{\bullet}xn$ - integral power, x to the n, x raised to the n power.

1.10.0 $\curlywedge x\underline{\mathrm{u}}x$ - function definition, the function taking x to the value $\underline{\mathrm{u}}x$.

1.14.0 The $x\underline{\mathrm{u}}x$ - unique description, the unique x such that $\underline{\mathrm{u}}x$ is true.

1.18 $(R : S)$ - composition of relations, the set of all ordered pais (x, z) such that for some y, the ordered pair (x, y) is in R and the ordered pair (y, z) is in S.

1.70 $\sup x \in A\underline{\mathrm{u}}x$ - supremum, sumpremum as x runs over A of $\underline{\mathrm{u}}x$.

1.14.0 $(\mathrm{prt}'\, x \equiv \mathrm{The}\, a \in \mathrm{rl} \bigvee b \in \mathrm{rf}(x = a + \mathrm{i} \cdot b))$.

The definiens is using a notation for unique descriptions, 'The $x\underline{\mathrm{u}}x$' , the unique x such that $\underline{\mathrm{u}}x$ is true. In this case, a special form of the description is used, 'The $x \in A\underline{\mathrm{u}}x$', which is the unique x in A for which $\underline{\mathrm{u}}x$ is true. This is equivalent to

$$\mathrm{The}\, x((x \in A) \wedge \underline{\mathrm{u}}x) \ .$$

A definition for this term is given in Morse's book. If there is no unique x such that $\underline{\mathrm{p}}x$ is true, then the $x\underline{\mathrm{p}}x$ is equal to U.

In 1.62.3, the notation 'sngl x' appears. This is a variation of the notion of sng x, singleton x. While sng x is equal to 0 in the case that x is a proper class, sngl x is equal to U in such case.

The remark after 7.46 uses the notation '$\underline{\mathrm{kf}}$' without any earlier explanatioin. This is Morse's notation for the standard metric on the complex plane. In MI, he uses a similar notation '$\underline{\mathrm{rf}}$' for the standard metric on the real line.

SI: Other Issues

In the proof of 7.74, the notation 'pwr G' occurs. This is Morse's notation for the cardinality of the set G. In the case at hand, the cardinality is equal to ω which is the cardinality of the set of natural numbers. (In fact, ω is the set of natural numbers.)

In the proof of 7.80.1, reference is made to 3.18A which is not part of these notes. This reference may be to a theorem in a set of Morse's notes from a course in analysis. That theorem is labelled 3.18* and states

$$\begin{aligned}&(\bigwedge x(\underline{\mathrm{u}}x \to \underline{\mathrm{v}}x \in \mathrm{rl}) \to \\ &\quad \sup x; (\underline{\mathrm{u}}x \wedge \underline{\mathrm{w}}x)\underline{\mathrm{v}}x \leq \sup x; \underline{\mathrm{u}}x\underline{\mathrm{v}}x \wedge \\ &\quad \inf x; \underline{\mathrm{u}}x\underline{\mathrm{v}}x \leq \inf x; (\underline{\mathrm{u}}x \wedge \underline{\mathrm{w}}x)\underline{\mathrm{v}}x)\end{aligned}$$

The Background Notation which follows was a compilation put together by Morse for his classes.

Background Notation

All of the notations below should be either familiar, or obvious, or at least fairly easy to figure out. Some are explicitly defined in the lecture notes.

All of the formulas below can be taken as theorems; some in fact as definitions.

Notations

.0 $((p \wedge q) \leftrightarrow (p \,\text{and}\, q))$

.1 $((p \vee q) \leftrightarrow (p \,\text{or}\, q))$

.2 $(\sim p \leftrightarrow \text{Not } p)$

.3 $(\bigwedge x \underline{\mathrm{u}}x \leftrightarrow \text{For each } x, \underline{\mathrm{u}}x)$

.4 $(\bigvee x \underline{\mathrm{u}}x \leftrightarrow \text{For some } x, \underline{\mathrm{u}}x)$

.5 $(\bigwedge x \in A \underline{\mathrm{u}}x \leftrightarrow \bigwedge x (x \in A \rightarrow \underline{\mathrm{u}}x))$

.6 $(\bigvee x \in A \underline{\mathrm{u}}x \leftrightarrow \bigvee x (x \in A \wedge \underline{\mathrm{u}}x))$

.7 $((x \ni y) \leftrightarrow (y \in x))$

.8 $((x \backsim\in y) \leftrightarrow \sim(x \in y))$

.9 $((x \backsim\subset y) \leftrightarrow \sim(x \subset y))$

.10 $((x \subset\cdot\, y) \leftrightarrow (x \subset y \wedge x \neq y))$

.11 $((xy) = (x \cap y))$

.12 $(x \backsim y = x{\sim}y = (x)(\sim y) = x \cap \sim y)$

'$x \backsim y$' is used in lieu of '$x{\sim}y$' for technical reasons in expressions such as '$\bigwedge x \in A \backsim B \underline{\mathrm{u}}x$'.

.13 $(\bigwedge x \in A \underline{\mathrm{u}}x = \text{The intersection as } x \text{ runs over } A, \text{ of } \underline{\mathrm{u}}x)$

.14 $(\bigvee x \in A \underline{\mathrm{u}}x = \text{The union as } x \text{ runs over } A, \text{ of } \underline{\mathrm{u}}x)$

.15 $(\mathsf{E}\, x \underline{\mathrm{u}}x = \{x : \underline{\mathrm{u}}x\} = \text{The set of points } x \text{ such that } \underline{\mathrm{u}}x)$

.16 $(\nabla A = \bigvee x \in A x = \text{The union of } A)$

.17 $(\sqcap A = \bigwedge x \in A x = \text{The intersection of } A)$

.18 $(\mathrm{U} = \text{The universe} = \sim 0)$

.19 $(0 = \text{The empty set} = \sim \mathrm{U})$

.20 $(\nabla \mathrm{U} = \mathrm{U} = \sqcap 0)$

.21 $(\sqcap \mathrm{U} = 0 = \nabla 0)$

.22 $(\mathrm{sb}\, A = \mathsf{E}\, x (x \subset A))$

.23 $(\mathrm{sp}\, A = \mathsf{E}\, x (x \supset A))$

.24 $(x \text{ is a point} \leftrightarrow \bigvee y (x \in y) \leftrightarrow x \in \mathrm{U})$

.25 $\sim(\mathrm{U} \text{ is a point})$

.26 $(\mathrm{sng}\, x = \mathsf{E}\, y (y = x))$

.27 $(x \backsim\in \mathrm{U} \rightarrow \mathrm{sng}\, x = 0)$

.28 $(\text{singleton is}\, A \leftrightarrow \bigvee x \in \mathrm{U} (A = \mathrm{sng}\, x))$

.29 $(\mathsf{E}\, x, y \underline{\mathrm{u}}' xy = \mathsf{E}\, z \bigvee x \bigvee y (z = x, y \wedge \underline{\mathrm{u}}' xy))$

.30 $(\text{ordered pair is}\, p \leftrightarrow \bigvee x \bigvee y (p = x, y))$

.31 $(\text{relation is}\, R \leftrightarrow \bigwedge p \in R \,\text{ordered pair is}\, p)$

.32 $(\mathrm{vs}\, Rx = \text{The vertical section of } R \text{ at } x = \mathsf{E}\, y (x, y \in R))$

.33 $(\mathrm{hs}\, Ry = \text{The horizontal section of } R \text{ at } y = \mathsf{E}\, x (x, y \in R))$

.34 $({}_{*}RA = \mathsf{E}\, y \bigvee x \in A (x, y \in R))$

.35 $({}^{*}RB = \mathsf{E}\, y \bigvee y \in B (x, y \in R))$

.36 $(\mathrm{dmn}\, R = \text{The domain of } R = \mathsf{E}\, x \bigvee y (x, y \in R))$

.37 $(\mathrm{rng}\, R = \text{The range of } R = \mathsf{E}\, y \bigvee x (x, y \in R))$

.38 $(\mathrm{inv}\, R = \text{The inverse of } R = \mathsf{E}\, x, y (y, x \in R))$

.39 $(\text{function is } f \leftrightarrow \text{relation is } f \wedge \bigwedge x \in \operatorname{dmn} f \text{ singleton is vs } fx)$
.40 $(\text{univalent is } f \leftrightarrow \text{function is } f \wedge \text{function is inv } f)$
.41 $(\text{Upon } A = \mathsf{E}\, f(\text{function is } f \wedge \operatorname{dmn} f \subset A))$
.42 $(\text{On } A = \mathsf{E}\, f(\text{function is } f \wedge \operatorname{dmn} f = A))$
.43 $(\text{To } B = \mathsf{E}\, f(\text{function is } f \wedge \operatorname{rng} f \subset B))$
.44 $(\text{Onto } B = \mathsf{E}\, f(\text{function is } f \wedge \operatorname{rng} f = B))$
.45 $(\text{Uto } B = \mathsf{E}\, f(\text{univalent is } f \wedge \operatorname{rng} f \subset B))$
.46 $(\text{Uonto } B = \mathsf{E}\, f(\text{univalent is } f \wedge \operatorname{rng} f = B))$
.47 $(\boldsymbol{.}\, fx = \sqcap \text{ vs } fx)$
.48 $(\measuredangle x \in A\underline{u}x = \mathsf{E}\, x, y(x \in A \wedge y = \underline{u}x))$
.49 $(f = \measuredangle x \in A\underline{u}x \rightarrow \text{function is } f \wedge \operatorname{dmn} f = \mathsf{E}\, x \in A(\underline{u}x \in \mathrm{U}) \wedge \bigwedge x \in \operatorname{dmn} f(\boldsymbol{.}\, fx = \underline{u}x))$
.50 $(\text{function is } f \rightarrow \bigwedge x \backsim\in \operatorname{dmn} f(\boldsymbol{.}\, fx = \mathrm{U}))$
.51 $(\omega = \text{The set of natural numbers})$
.52 $(1 = \operatorname{sng} 0 \wedge 2 = 1 \cup \operatorname{sng} 1 = \{01\} \wedge 3 = 2 \cup \operatorname{sng} 2 = \{012\} \wedge \bigwedge n \in \omega(n + 1 = n \cup \operatorname{sng} n))$
.53 $(\mathrm{fnt} = \mathsf{E}\, x \text{ finite is } x = \mathsf{E}\, x \bigvee n \in \omega \bigvee f(f \in \text{On } x \text{ Uonto } n))$
.54 $(\mathrm{cbl} = \mathsf{E}\, x \text{ countable is } x = \mathsf{E}\, x \bigvee f(f \in \text{On } x \text{ Uto } \omega))$
.55 $(\operatorname{sqnc} A = \mathsf{E}\, f(\text{sequence is } f \wedge \operatorname{rng} f \subset A) = \text{On } \omega \text{ To } A)$
.56 $(x \backsim\in \mathrm{rf} \rightarrow |x| = \infty)$
.57 $(\mathrm{rl} = \mathsf{E}\, x(-\infty \leq x \leq \infty))$
.58 $(\mathrm{rf} = \mathsf{E}\, x(-\infty < x < \infty))$
.59 $(\mathrm{rfp} = \mathsf{E}\, x(0 < x < \infty))$
.60 $(\sup x \in 0\underline{u}x = -\infty \wedge \inf x \in 0\underline{u}x = \infty \wedge \sum x \in 0\underline{u}x = 0)$
.61 $(0 \leq y \rightarrow \sqrt{y} = \sup x \in \mathrm{rf}(x \cdot x \leq y) \wedge 0 \leq \sqrt{y} \wedge \sqrt{y} \cdot \sqrt{y} = y)$
.62 $(\sim \bigwedge x \in A(\underline{u}x \in \mathrm{rl}) \rightarrow \sup x \in A\underline{u}x = \inf x \in A\underline{u}x = \mathrm{U})$
.63 $(\text{big } n\underline{u}n \leftrightarrow \bigvee N \in \omega \bigwedge n \in \omega(N \leq n \rightarrow \underline{u}n) \leftrightarrow \bigvee N \in \omega \bigwedge n \in \omega \backsim N\underline{u}n)$
.64 $(\text{lin } n\underline{u}n = \text{The sequential limit of } \underline{u}n \text{ with respect to } n)$
.65 $(x + y \cdot z/w = x + ((y \cdot z)/w))$
.66 $(\operatorname{rct} AB = \mathsf{E}\, x, y(x \in A \wedge y \in B))$
.67 $(\operatorname{sqr} A = \operatorname{rct} AA)$
.68 $((\rho \text{ metrizes } A) \leftrightarrow (\rho \in \text{On sqr } A \text{ To rf} \wedge$
$\bigwedge x \in A \bigwedge y \in A \bigwedge z \in A(0 = \boldsymbol{.}\,\rho(x, x) \leq \boldsymbol{.}\,\rho(x, z) \leq \boldsymbol{.}\,\rho(x, y) + \boldsymbol{.}\,\rho(y, z))))$
.69 $(\text{space } \rho = \sqcap \mathsf{E}\, A(\rho \text{ metrizes } A))$
.70 $(\text{metric} = \mathsf{E}\, \rho(\rho \text{ metrizes space } \rho))$
.71 $(\underline{\mathrm{rf}} = \text{The standard metric on rf} = \measuredangle x, y \in \text{sqr rf } |x - y|)$
.72 $(\operatorname{sr} \rho xy = \mathsf{E}\, z(\boldsymbol{.}\,\rho(x, z) < y))$
.73 $(\operatorname{intr} \rho A = \text{The interior of } A \text{ under } \rho = \mathsf{E}\, x \bigvee y \in \mathrm{rfp}(\operatorname{sr} \rho xy \subset A))$
.74 $(\operatorname{clsr} \rho A = \text{The closure of } A \text{ under } \rho = \mathsf{E}\, x \bigwedge y \in \mathrm{rfp}(A \cap \operatorname{sr} \rho xy \neq 0))$
.75 $(\operatorname{diam} \rho A = \operatorname{Sup}(\operatorname{sng} 0 \cup \bigvee x \in A \bigvee y \in A\{\boldsymbol{.}\,\rho(x, y)\}))$
.76 $(\text{bounded } \rho = \mathsf{E}\, A(\operatorname{diam} \rho A < \infty))$
.77 $(\operatorname{dist} \rho AB = \inf x \in A \inf y \in B \,\boldsymbol{.}\,\rho(x, y))$
.78 $(\text{open } \rho = \mathsf{E}\, A(\rho \in \text{metric} \wedge A = \operatorname{intr} \rho A))$
.79 $(\text{closed } \rho = \mathsf{E}\, A(\rho \in \text{metric} \wedge A = \operatorname{clsr} \rho A))$
.80 $(\underline{2}\, n = 1/^{\boldsymbol{.}} 2n)$
.81 $(\mathrm{ndx} = \text{index} = \mathsf{E}\, N \in \operatorname{sqnc} \omega \bigwedge n \in \omega(\boldsymbol{.}\, Nn < \boldsymbol{.}\, N(n + 1)))$
.82 $(\operatorname{sbqnc} p = \mathsf{E}\, q(p \in \operatorname{sqnc} \mathrm{U} \wedge \bigvee N \in \mathrm{ndx} \bigwedge n \in \omega(\boldsymbol{.}\, qn = \boldsymbol{.}\, p \,\boldsymbol{.}\, Nn)))$
.83 $(\operatorname{strc} fA = \measuredangle x \in A \,\boldsymbol{.}\, fx)$
.84 $(\underline{\underline{\mathrm{lin}}}\, n\underline{u}n = \text{lin } m \inf n \in \omega \backsim m\underline{u}n)$
.85 $(\overline{\overline{\mathrm{lin}}}\, n\underline{u}n = \text{lin } m \sup n \in \omega \backsim m\underline{u}n)$

.86 (run is $R \leftrightarrow$ relation is $R \wedge R \neq 0 \wedge$

$\bigwedge x \in \text{dmn}\, R \bigwedge y \in \text{dmn}\, R \bigvee z \in \text{dmn}\, R(\text{vs}\, Rz \subset \text{vs}\, Rx \cap \text{vs}\, Ry))$

.87 (Nb $rp = \mathsf{E}\, x(p = \infty \wedge x \geq 1/r) \cup \mathsf{E}\, x(p \in \text{rf} \wedge |x - p| \leq r) \cup \mathsf{E}\, x(p = -\infty \wedge x \leq -1/r))$

.88 (lm $xR\underline{\text{u}}x = \sqcap \mathsf{E}\, p(\bigwedge r \in \text{rfp} \bigvee y \in \text{dmn}\, R \bigwedge x \in \text{vs}\, Ry(\underline{\text{u}}x \in \text{Nb}\, rp) \wedge \text{run is}\, R))$

.89 (indexrun $R = \mathsf{E}\, x, y(\text{run is}\, R \wedge x \in \text{dmn}\, R \wedge y \in \text{dmn}\, R \wedge \text{vs}\, Rx \supset \text{vs}\, Ry))$

.90 ($\overline{\text{lm}}\, xR\underline{\text{u}}x = \text{lm}\, y\, \text{indexrun}\, R \sup x \in \text{vs}\, Ry\underline{\text{u}}x)$

.91 ($\underline{\text{lm}}\, xR\underline{\text{u}}x = \text{lm}\, y\, \text{indexrun}\, R \inf x \in \text{vs}\, Ry\underline{\text{u}}x)$

.92 (Cauchy $\rho =$ The set of Cauchy sequences under the metric ρ)

.93 (cvg $\rho x = \mathsf{E}\, S \in \text{sqnc}\, \rho(\rho \in \text{metric} \wedge \text{lin}\, n \,.\, \rho(.\, Sn, x) = 0))$

.94 (Complete $\rho = \mathsf{E}\, A(\rho \in \text{metric} \wedge \bigwedge S \in \text{Cauchy}\, \rho \cap \text{To}\, A \bigvee x \in A(S \in \text{cvg}\, \rho x)))$

.95 (Continuous $\rho\zeta = \mathsf{E}\, f(\text{function is}\, f \wedge \rho \in \text{metric} \wedge \zeta \in \text{metric} \wedge \bigwedge x \in \text{open}\, \zeta(*fx \in \text{open}\, \rho)))$

Table of Contents

Real Numbers

R.1 Definitions

.0 $((x — y) \equiv (x + - y))$
.1 $(\mathrm{rl} \equiv \mathsf{E}\, x(\mathrm{Sup}\,\mathrm{sng}\, x \neq \mathrm{U}))$
.2 $(\mathrm{The\,reals} \equiv \mathrm{rl})$
.3 $(\mathrm{rf} \equiv \mathsf{E}\, x \in \mathrm{rl}(x — x = 0))$
.4 $(\mathrm{rp} \equiv \mathsf{E}\, x \in \mathrm{rl} \backsim 1(\mathrm{Sup}(\mathrm{sng}\, x \cup 1) \neq 0))$
.5 $(\mathrm{rfp} \equiv (\mathrm{rf} \cap \mathrm{rp}))$
.6 $((x < y) \equiv (x, y, \in \mathrm{rl} \wedge y — x \in \mathrm{rp}))$
.7 $((x \leq y) \equiv (x < y \vee x = y \in \mathrm{rl}))$
.8 $((x > y) \equiv (y < x))$
.9 $((x \geq y) \equiv (y \leq x))$
.10 $(\infty \equiv \mathrm{Sup}\,\mathrm{rl})$

R.2 Postulates

.0 $(0 \in \omega \subset \mathrm{rf} \in \mathrm{U})$
.1 $(x \in \omega \to x \cup \mathrm{sng}\, x = x + 1 \in \omega)$
.2 $(0 \in S \subset \omega \wedge \bigwedge n \in S(n + 1 \in S) \to S = \omega)$

R.3 Postulates

.0 $(x + y \neq \mathrm{U} \to x \in \mathrm{U} \wedge x + y \in \mathrm{U})$
.1 $(x \cdot y \neq \mathrm{U} \to x \in \mathrm{U} \wedge x \cdot y \in \mathrm{U})$
.2 $(1/x \neq \mathrm{U} \to x \in \mathrm{U} \wedge 1/x \in \mathrm{U})$

R.4 Postulates

.0 $(x \in \mathrm{rl} \to -x \in \mathrm{rl})$
.1 $(x \in \mathrm{rl} \to 1/x \in \mathrm{rl} \leftrightarrow x \neq 0)$
.2 $(0 \neq x \in \mathrm{rl} \to x \in \mathrm{rp} \vee -x \in \mathrm{rp})$
.3 $(x \in \mathrm{rl} \to x \backsim\in \mathrm{rf} \leftrightarrow x — x = \mathrm{U}$
$\leftrightarrow 1/x = 0$
$\leftrightarrow x = \infty \vee x = -\infty)$
.4 $(\mathrm{Sup}\, A \neq \mathrm{U} \leftrightarrow A \subset \mathrm{rl} \leftrightarrow \mathrm{Sup}\, A \in \mathrm{rl})$

R.5 Postulates

.0 $(x, y, \in \mathrm{rf} \to x + y \in \mathrm{rf} \wedge x \cdot y \in \mathrm{rf})$
.1 $(x, y, \in \mathrm{rp} \to x + y \in \mathrm{rp} \wedge x \cdot y \in \mathrm{rp})$
.2 $(x \in \mathrm{rl} \wedge x \cdot y \in \mathrm{rl} \to y \in \mathrm{rl})$
.3 $(x \cdot y = 0 \to x = 0 \vee y = 0)$

R.6 Postulates

.0 $(-x = -1 \cdot x)$
.1 $(x/y = x \cdot (1/y))$
.2 $(x \in \mathrm{rf} \wedge y \in \mathrm{rl} \backsim \mathrm{rf} \to x + y = y)$

R.7 Postulates

.0 $(x + y = y + x)$
.1 $(x \cdot y = y \cdot x)$
.2 $(x + (y + z) = x + y + z = (x + y) + z)$

.3 $(x \cdot (y \cdot z) = x \cdot y \cdot z = (x \cdot y) \cdot z)$
.4 $(z \in \mathrm{rf} \rightarrow z \cdot (x + y) = z \cdot x + z \cdot y)$
.5 $(x \in \mathrm{rl} \rightarrow 1 \cdot x = x + 0 = x)$
.6 $(0 \neq x \in \mathrm{rf} \rightarrow x/x = 1)$
.7 $(A \subset \mathrm{rl} \wedge t \in \mathrm{rl} \rightarrow \bigwedge x \in A(y \leq t) \leftrightarrow \mathrm{Sup}\, A \leq t)$

R.8 Definitions
.0 $(\omega' \equiv \exists\, n(n \in \omega \vee -n \in \omega))$
.1 $(\text{The integers} \equiv \omega')$

R.9 Postulates
.0 $(x \in \mathrm{rl} \rightarrow \dot{}x0 = 1)$
.1 $(x \in \mathrm{rl} \wedge n \in \omega \rightarrow \dot{}x(n+1) = x \cdot \dot{}xn)$
.2 $(n \in \omega' \backsim \omega \rightarrow \dot{}xn = \dot{}(1/x) - n)$
.3 $(\dot{}xn \neq \mathrm{U} \rightarrow n \in \omega' \wedge x \in \mathrm{U})$

Definition
R.10 $(\mathrm{noz}\, x \equiv (x = 0 \vee x))$

R.11 Postulates
.0 $(x \in \mathrm{rl} \wedge 0 \neq y \in \mathrm{To\,rl} \rightarrow x + y = \mathrm{noz} \measuredangle t(x + \bullet\, yt) \wedge$
$x \cdot y = \mathrm{noz} \measuredangle t(x \cdot \bullet\, yt))$
.1 $(x, y, \in \mathrm{To\,rl} \backsim 1 \rightarrow x + y = \mathrm{noz} \measuredangle t(\bullet\, xt + \bullet\, yt) \wedge$
$x \cdot y = \mathrm{noz} \measuredangle t(\bullet\, xt \cdot \bullet\, yt))$

R.12 Definitions
.0 $(\sup x \in A\underline{\mathrm{u}}x \equiv \mathrm{Sup} \bigvee x \in A\{\underline{\mathrm{u}}x\})$
.1 $(\inf x \in A\underline{\mathrm{u}}x \equiv -\sup x \in A - \underline{\mathrm{u}}x)$

Definition
R.13 $(\mathrm{Inf}\, A \equiv \inf x \in Ax)$

R.14 Theorems
.0 $(x \in \mathrm{rf} \rightarrow 0 \cdot x = 0)$
Proof:
$(0 = x - x = x \cdot 1 + x \cdot -1 = x \cdot (1 - 1) = x \cdot 0 = 0 \cdot x)$
.1 $(x \in \mathrm{rf} \rightarrow -x \in \mathrm{rf})$
Proof: R.4.0 and R.6.2
.2 $(-1 \cdot -1 = 1)$
Proof:
$(1 + -1 = 0 \wedge -1 \cdot 1 + -1 \cdot -1 = 0 \wedge -1 + -1 \cdot -1 = 0 \wedge -1 \cdot -1 = 1)$
.3 $(1 \in \mathrm{rp})$
Proof: Using R.4.2, R.5.1, and R.14.2
$(1 \neq 0 \rightarrow 1 \in \mathrm{rp} \vee -1 \in \mathrm{rp}$
$\rightarrow 1 \in \mathrm{rp} \vee -1 \cdot -1 \in \mathrm{rp}$
$\rightarrow 1 \in \mathrm{rp} \vee 1 \in \mathrm{rp}$
$\rightarrow 1 \in \mathrm{rp})$
.4 $(x \in \mathrm{rl} \backsim \mathrm{rf} \rightarrow x + y \backsim\in \mathrm{rf})$
.5 $(\infty \in \mathrm{rl} \wedge -\infty \in \mathrm{rl})$

.6 $(\infty + \infty = \infty)$

Hint: $(\infty \in \text{rp} \vee -\infty \in \text{rp})$

$(\infty + \infty \in \text{rl} \backsim \text{rf} \wedge \infty + \infty \neq -\infty)$

.7 $(0 \cdot \infty = \text{U})$

Proof:

$(\text{U} = 0 \cdot \text{U} = 0 \cdot (\infty - \infty) = 0 \cdot (\infty + -1 \cdot \infty) = 0 \cdot \infty + 0 \cdot \infty = 0 \cdot (\infty + \infty) = 0 \cdot \infty)$

.8 $(x \in \text{rl} \backsim \text{rf} \rightarrow 0 \cdot x = \text{U})$

.9 $(x \in \text{rl} \rightarrow x \in \text{rf} \leftrightarrow x - x = 0$

$\leftrightarrow 0 \cdot x = 0$

$\leftrightarrow 0 \cdot x \neq \text{U}$

$\leftrightarrow x - x \neq \text{U})$

.10 $(x, y, \in \text{rl} \rightarrow x + y \in \text{rf} \leftrightarrow x, y, \in \text{rf} \leftrightarrow x \cdot y \in \text{rf})$

.11 $(x \in \text{rl} \leftrightarrow -x \in \text{rl})$

.12 $(x \in \text{rf} \leftrightarrow -x \in \text{rf})$

Remark. Inequalities behave reasonably, for example

$(a, b, \in \text{rl} \wedge a + b > c \rightarrow \bigvee a' < a \bigvee b' < b(a' + b' > c))$

.13 $(\text{rl} = \mathsf{E}\, x(-\infty \leq x \leq \infty))$

.14 $(\text{rf} = \mathsf{E}\, x(-\infty < x < \infty))$

.15 $(-\infty < x \rightarrow x + \infty = \infty)$

.16 $(x < \infty \rightarrow x - \infty = -\infty)$

.17 $(0 < x \rightarrow x \cdot \infty = \infty \wedge x \cdot -\infty = -\infty)$

.18 $(x < 0 \rightarrow x \cdot \infty = -\infty \wedge x \cdot -\infty = \infty)$

To facilitate computation

R.15 $(x \in \text{rl} \vee y \in \text{rl} \rightarrow 0 + x + y = x + y \wedge 1 \cdot x \cdot y = x \cdot y)$

Chapter 1: Summation

1.0 Definitions

.0 $(\operatorname{ps} x \equiv \operatorname{Inf} \mathsf{E}\, t(x \leq t \geq 0))$

.1 $(\operatorname{ng} x \equiv \operatorname{ps} - x)$

.2 $(\operatorname{ril} x \equiv (x \in \mathrm{rl} \vee x = \mathrm{U}))$

.3 $(\operatorname{rilp} x \equiv (0 \leq x \vee x = \mathrm{U})$

.4 $(|x| \equiv (\operatorname{ps} x + \operatorname{ng} x))$

1.1 Theorems

.0 $(0 \leq x \rightarrow \operatorname{ps} x = x)$

.1 $(x \leq 0 \rightarrow \operatorname{ps} x = 0)$

.2 $(x \backsim\in \mathrm{rl} \rightarrow \operatorname{ps} x = \infty = \operatorname{ng} x)$

.3 $(0 \leq \operatorname{ps} x \wedge 0 \leq \operatorname{ng} x)$

.4 $(x \backsim\in \mathrm{rl} \rightarrow \operatorname{ps} x \text{ --- } \operatorname{ng} x = \mathrm{U})$

.5 $(\operatorname{ril} x \rightarrow x = \operatorname{ps} x + \operatorname{ng} x)$

.6 $(|x| = \operatorname{ps} x + \operatorname{ng} x)$

.7 $(\operatorname{ps}(x + y) + \operatorname{ng} x + \operatorname{ng} y = \operatorname{ng}(x + y) + \operatorname{ps} x + \operatorname{ps} y)$

.8 $(\operatorname{ng} - x = \operatorname{ps} x)$

.9 $(0 \leq \operatorname{ps}(x + y) \leq \operatorname{ps} x + \operatorname{ps} y)$

.10 $(0 \leq \operatorname{ng}(x + y) \leq \operatorname{ng} x + \operatorname{ng} y)$

.11 $(0 < c < \infty \rightarrow \operatorname{ps}(c \cdot x) = c \cdot \operatorname{ps} x \wedge \operatorname{ng}(c \cdot x) = c \cdot \operatorname{ng} x)$

Hint: Use R.5.2

.12 $(\operatorname{ril} a \wedge \operatorname{ril} b \rightarrow \operatorname{ril}(a + b) \wedge \operatorname{ril}(a \cdot b))$

1.2 Postulates

.0 $(\bigwedge x \in A(\underline{\mathrm{u}}x = \underline{\mathrm{v}}x) \rightarrow \sum x \in A\underline{\mathrm{u}}x = \sum x \in A\underline{\mathrm{v}}x)$

.1 $(\sum x \in 0\underline{\mathrm{u}}x = 0)$

.2 $(y \in \mathrm{U} \wedge \underline{\mathrm{u}}y \in \mathrm{rl} \rightarrow \sum x \in \operatorname{sng} y\underline{\mathrm{u}}x = \underline{\mathrm{u}}y)$

.3 $(y \in \mathrm{U} \wedge \underline{\mathrm{u}}y \backsim\in \mathrm{rl} \rightarrow \sum x \in \operatorname{sng} y\underline{\mathrm{u}}x = \mathrm{U})$

.4 $(A \cap B = 0 \rightarrow \sum x \in A \cup B\underline{\mathrm{u}}x = \sum x \in A\underline{\mathrm{u}}x + \sum x \in B\underline{\mathrm{u}}x)$

.5 $(\bigwedge x \in A(\underline{\mathrm{u}}x \geq 0) \rightarrow \sum x \in A\underline{\mathrm{u}}x = \sup \alpha \in \mathrm{fnt} \cap \mathrm{sb}\, A \sum x \in \alpha\underline{\mathrm{u}}x)$

.6 $(\sum x \in A\underline{\mathrm{u}}x = \sum x \in A \operatorname{ps} \underline{\mathrm{u}}x \text{ --- } \sum x \in A \operatorname{ng} \underline{\mathrm{u}}x)$

Definition

1.3 $(\sum x\underline{\mathrm{u}}x \equiv \sum x \in \mathrm{U}\underline{\mathrm{u}}x)$

1.4 Theorems

.0 $\operatorname{ril} \sum x \in A\underline{\mathrm{u}}x$

.1 $(\sum x \in A\underline{\mathrm{u}}x \in \mathrm{rl} \rightarrow \bigwedge x(\underline{\mathrm{u}}x \in \mathrm{rl}))$

.2 $(\sum x \in A\underline{\mathrm{u}}x \in \mathrm{rf} \rightarrow \bigwedge x(\underline{\mathrm{u}}x \in \mathrm{rf}))$

Summation by finite partition

1.5 $(B \in \mathrm{fnt} \wedge V = \bigvee y \in B\underline{\mathrm{v}}y \wedge \bigwedge y, z, \in B(y \neq z \rightarrow \underline{\mathrm{v}}y \cap \underline{\mathrm{v}}z = 0) \rightarrow \sum x \in V\, \underline{\mathrm{u}}x = \sum y \in B \sum x \in \underline{\mathrm{v}}y\underline{\mathrm{u}}x)$

Proof: Hint: Induction on $\operatorname{pwr} B$

Theorems

1.6 $(B \in \mathrm{fnt} \rightarrow \sum x \in B(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) = \sum x \in B\underline{\mathrm{u}}x + \sum x \in B\underline{\mathrm{v}}x)$

1.7 $(\sum x \in A0 = 0)$

1.8 $(\sum x \in A - \underline{u}x = -\sum x \in A\underline{u}x)$
1.9 $(0 \neq c \in \mathrm{rf} \to \sum x \in A(c \cdot \underline{u}x) = c \cdot \sum x \in A\underline{u}x)$
1.10 $(\bigwedge x \in A(\underline{u}x \geq 0) \to 0 \leq \sum x \in A\underline{u}x)$
1.11 $(\bigwedge x \in A(\underline{u}x \geq 0) \to \sum x \in A\underline{u}x = 0 \leftrightarrow \bigwedge x \in A(\underline{u}x = 0))$

1.12 Theorems
.0 $(\sum x \in A\underline{u}x + \sum x \in B\underline{u}x = \sum x \in A \cup B\underline{u}x + \sum x \in A \cap B\underline{u}x)$
.1 $(\bigwedge x \in A(\underline{u}x \geq 0) \to \sum x \in A \cup B\underline{u}x \leq \sum x \in A\underline{u}x + \sum x \in B\underline{u}x)$

Theorems
1.13 $(\bigwedge x \in A(\underline{u}x \geq 0 \wedge \underline{v}x \geq 0) \to \sum x \in A(\underline{u}x + \underline{v}x) = \sum x \in A\underline{u}x + \sum x \in A\underline{v}x)$
1.14 $(\bigwedge x \in A(0 \leq \underline{u}x \leq \underline{v}x) \to 0 \leq \sum x \in A\underline{u}x \leq \sum x \in A\underline{v}x)$

Lemma
1.15 $(-\infty < \sum x \in A\underline{u}x + \sum x \in A\underline{v}x = s \to \sum x \in A(\underline{u}x + \underline{v}x) = s)$
Proof:
Evidently
$(\sum x \in A\,\mathrm{ng}\,\underline{u}x + \sum x \in A\,\mathrm{ng}\,\underline{v}x < \infty)$
and it is clear from 1.1.10, 1.14, and 1.13 that
.0 $(0 \leq \sum x \in A\,\mathrm{ng}(\underline{u}x + \underline{v}x) \leq \sum x \in A(\mathrm{ng}\,\underline{u}x + \mathrm{ng}\,\underline{v}x) \leq \sum x \in A\,\mathrm{ng}\,\underline{u}x + \sum x \in A\,\mathrm{ng}\,\underline{v}x < \infty)$
Next use 1.1.7 and 1.13 in checking
$(\sum x \in A\,\mathrm{ps}(\underline{u}x + \underline{v}x) + \sum x \in A\,\mathrm{ng}\,\underline{u}x + \sum x \in A\,\mathrm{ng}\,\underline{v}x$
$= \sum x \in A\,\mathrm{ng}(\underline{u}x + \underline{v}x) + \sum x \in A\,\mathrm{ps}\,\underline{u}x + \sum x \in A\,\mathrm{ps}\,\underline{v}x)$.
From this, .0, and R.15 we infer
$(\sum x \in A\,\mathrm{ps}(\underline{u}x + \underline{v}x) - \sum x \in A\,\mathrm{ng}(\underline{u}x + \underline{v}x)$
$= \sum x \in A\,\mathrm{ps}\,\underline{u}x - \sum x \in A\,\mathrm{ng}\,\underline{u}x + \sum x \in A\,\mathrm{ps}\,\underline{v}x - \sum x \in A\,\mathrm{ng}\,\underline{v}x)$.
Then apply 1.2.6.

An almost immediate consequence of 1.15 and 1.8 is
1.16 $(\sum x \in A\underline{u}x + \sum x \in A\underline{v}x \in \mathrm{rl} \to$
$\sum x \in A(\underline{u}x + \underline{v}x) = \sum x \in A\underline{u}x + \sum x \in A\underline{v}x)$
1.17 $(\bigwedge x \in A(\underline{u}x \leq \underline{v}x) \to \sum x \in A\underline{u}x \backsim< \sum x \in A\underline{v}x)$

1.18 Theorems
.0 $(0 \leq |x| \leq \infty)$
.1 $(x \in \mathrm{rf} \leftrightarrow 0 \leq |x| < \infty)$
.2 $(|x| = 0 \leftrightarrow x = 0)$
.3 $(|-x| = |x|)$
.4 $(|x + y| \leq |x| + |y|)$
.5 $(\mathrm{ril}\,x \wedge \mathrm{ril}\,y \to ||x| - |y|| \leq |x - y|)$
.6 $(x, y, \in \mathrm{rf} \to |x \cdot y| \leq |x| \cdot |y|)$
.7 $(0 \neq y \in \mathrm{rl} \to |1/y| = 1/|y|)$
.8 $(x \in \mathrm{rl} \to |x| = x \vee |x| = -x)$
.9 $(x \in \mathrm{rl} \to x \leq |x|)$

Theorems
1.19 $(|\sum x \in A\underline{u}x| \leq \sum x \in A|\underline{u}x|)$
1.20 $(\sum x \in A \in \mathrm{rf} \leftrightarrow \sum x \in A|\underline{u}x| < \infty)$

Definition

1.21 $((a \bullet b) \equiv (a \cdot b \wedge a, b, \neq 0))$

1.22 Theorems

.0 $(a, b, \neq 0 \rightarrow a \bullet b = a \cdot b)$

.1 $(0 < |a| < \infty \rightarrow a \bullet b = a \cdot b)$

.2 $(a \bullet b = b \bullet a)$

.3 $(a \bullet (b \bullet c) = (a \bullet b) \bullet c)$

.4 $(a \bullet 0 = 0 \bullet a = 0)$

.5 $(c \in \mathrm{rf} \rightarrow c \bullet (a + b) = c \bullet a + c \bullet b)$

.6 $(\mathrm{rilp}\, a \wedge \mathrm{rilp}\, b \wedge \mathrm{rilp}\, c \rightarrow c \bullet (a + b) = c \bullet a + c \bullet b)$

.7 $(\mathrm{ril}\, r \wedge r \bullet c \in \mathrm{rl} \rightarrow r = 0 \vee c \in \mathrm{rl})$

.8 $(a, b, \in \mathrm{rl} \rightarrow a \bullet b \in \mathrm{rl})$

.9 $(a, b, \geq 0 \rightarrow a \bullet b \geq 0)$

.10 $(\mathrm{ril}\, a \wedge \mathrm{ril}\, b \rightarrow \mathrm{ril}(a \bullet b) \wedge \mathrm{ril}(a + b) \wedge \mathrm{ril}(a \cdot b))$

.11 $(\mathrm{rilp}\, a \wedge \mathrm{rilp}\, b \rightarrow \mathrm{rilp}(a \bullet b) \wedge \mathrm{rilp}(a + b))$

.12 $(\mathrm{ril}\, c \wedge a \bullet c \in \mathrm{rl} \wedge b \bullet c \in \mathrm{rl} \wedge a \bullet c + b \bullet c \in rl \rightarrow$
$a \bullet c + b \bullet c = (a + b) \bullet c)$

.13 $(0 \leq c \rightarrow \mathrm{ps}(c \bullet x) = c \bullet \mathrm{ps}\, x \wedge \mathrm{ng}(c \bullet x) = c \bullet \mathrm{ng}\, x)$

.14 $(a \leq b \wedge 0 \leq c \rightarrow c \bullet a \leq c \bullet b)$

.15 $(\mathrm{ril}\, x \rightarrow 0 + x = x \wedge 1 \bullet x = 1 \cdot x = x)$

1.23 Theorems

.0 $(\mathrm{ril}\, y \rightarrow |x \bullet y| = |x| \bullet |y|)$

.1 $(|x| \bullet |y| < \infty \rightarrow |x \bullet y| = |x| \bullet |y|)$

We repeat 1.4.0 in 1.24.0.

1.24 Theorems

.0 $\mathrm{ril} \sum x \in A \underline{\mathrm{u}} x$

.1 $(c \in \mathrm{rf} \rightarrow \sum x \in A(c \bullet \underline{\mathrm{u}} x) = c \bullet \sum x \in A \underline{\mathrm{u}} x)$

.2 $(\sum x \in A(1 \bullet \underline{\mathrm{u}} x) = 1 \bullet \sum x \in A \underline{\mathrm{u}} x = \sum x \in A \underline{\mathrm{u}} x)$

.3 $(\mathrm{ril}\, c \wedge \bigwedge x \in A\, \mathrm{rilp}\, \underline{\mathrm{u}} x \rightarrow \sum x \in A(c \bullet \underline{\mathrm{u}} x) = c \bullet \sum x \in A \underline{\mathrm{u}} x)$

.4 $(\bigwedge x \in A\, \mathrm{rilp}\, \underline{\mathrm{u}} x \wedge c \bullet \sum x \in A \underline{\mathrm{u}} x \in \mathrm{rl} \rightarrow \sum x \in A(c \bullet \underline{\mathrm{u}} x) = c \bullet \sum x \in A \underline{\mathrm{u}} x)$

.5 $(\bigwedge x \in A\, \mathrm{rilp}\, \underline{\mathrm{u}} x \wedge \sum x \in A(c \bullet \underline{\mathrm{u}} x) \in \mathrm{rl} \rightarrow \sum x \in A(c \bullet \underline{\mathrm{u}} x) = c \bullet \sum x \in A \underline{\mathrm{u}} x)$

.6 $(|c| \bullet \sum x \in A |\underline{\mathrm{u}} x| < \infty \rightarrow \sum x \in A(c \bullet \underline{\mathrm{u}} x) = c \bullet \sum x \in A \underline{\mathrm{u}} x \in \mathrm{rf})$

.7 $(\sum x \in A(c \bullet \underline{\mathrm{u}} x) \in \mathrm{rl} \wedge c \bullet \sum x \in A \underline{\mathrm{u}} x \in \mathrm{rl} \rightarrow \sum x \in A(c \bullet \underline{\mathrm{u}} x) = c \bullet \sum x \in A \underline{\mathrm{u}} x)$

With the help of .0, .4, .5, and 1.4 we see

1.25 $(\bigwedge x \in A \bigwedge y \in B\, \mathrm{rilp}\, \underline{\mathrm{v}}' xy \rightarrow \sum x \in A(\underline{\mathrm{u}} x \bullet \sum y \in B \underline{\mathrm{v}}' xy) = \sum x \in A \sum y \in B(\underline{\mathrm{u}} x \bullet \underline{\mathrm{v}}' xy))$

Definition

1.26 $(\mathrm{Cr}\, xy \equiv (1 \wedge x \in y))$

1.27 Theorems

.0 $(\mathrm{Cr}\, xy = 1 \leftrightarrow x \in y)$

.1 $(\mathrm{Cr}\, xy = 0 \leftrightarrow x\!\sim\, \in y)$

Theorem
1.28 $(\sum x \in A\underline{\mathrm{u}}x = \sum x(\mathrm{Cr}\, xA \bullet \underline{\mathrm{u}}x))$

Lemma
1.29 $(\bigwedge x, y(\underline{\mathrm{u}}'xy \geq 0) \wedge A \in \mathrm{fnt} \rightarrow \sum x \in A \sum y\underline{\mathrm{u}}'xy = \sum y \sum x \in A\underline{\mathrm{u}}'xy)$
Hint: use induction on pwr A .

Lemma
1.30 $(\bigwedge x, y(\underline{\mathrm{u}}'xy \geq 0) \rightarrow \sum x \sum y\underline{\mathrm{u}}'xy \leq \sum y \sum x\underline{\mathrm{u}}'xy)$

Theorems
1.31 $(\bigwedge x, y(\underline{\mathrm{u}}'xy \geq 0) \rightarrow \sum x \sum y\underline{\mathrm{u}}'xy = \sum y \sum x\underline{\mathrm{u}}'xy)$

Very useful is
1.32 $(\bigwedge x, y(\underline{\mathrm{v}}'xy \geq 0) \wedge S = \sum x(\underline{\mathrm{u}}x \bullet \sum y\underline{\mathrm{v}}'xy) \in \mathrm{rl} \rightarrow S = \sum y \sum x(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy))$
Proof:
$(\mathrm{rl} \ni S = \sum x(\underline{\mathrm{u}}x \bullet \sum y\underline{\mathrm{v}}'xy)$
$= \sum x\, \mathrm{ps}(\underline{\mathrm{u}}x \bullet \sum y\underline{\mathrm{v}}'xy) — \sum x\, \mathrm{ng}(\underline{\mathrm{u}}x \bullet \sum y\underline{\mathrm{v}}'xy)$
$= \sum x(\mathrm{ps}\, \underline{\mathrm{u}}x \bullet \sum y\underline{\mathrm{v}}'xy) — \sum x(\mathrm{ng}\, \underline{\mathrm{u}}x \bullet \sum y\underline{\mathrm{v}}'xy)$
$= \sum x \sum y(\mathrm{ps}\, \underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy) — \sum x \sum y(\mathrm{ng}\, \underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy)$
$= \sum y \sum x(\mathrm{ps}\, \underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy) — \sum y \sum x(\mathrm{ng}\, \underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy)$
$= \sum y(\sum x(\mathrm{ps}\, \underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy) — \sum x(\mathrm{ng}\, \underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy))$
$= \sum y \sum x(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy))$

A generaization of 1.32 is
1.33 $(\bigwedge x, y\, \mathrm{rilp}\, \underline{\mathrm{w}}'xy \wedge S = \sum x(\underline{\mathrm{u}}x \bullet \sum y\underline{\mathrm{w}}'xy) \in \mathrm{rl} \rightarrow S = \sum y \sum x(\underline{\mathrm{u}}x \bullet \underline{\mathrm{w}}'xy))$
Hint: Assume
$\bigwedge x, y(\underline{\mathrm{v}}'xy = |\underline{\mathrm{w}}'xy|)$.
Check that
$\bigwedge x, y, \in \mathrm{U}(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy = \underline{\mathrm{u}}x \bullet \underline{\mathrm{w}}'xy)$
and then make use of 1.25, 1.2.0, 1.3, and 1.32.

We now have almost at once
Summation by positive distribution
1.34 $(\bigwedge x \in A \bigwedge y \in B\, \mathrm{rilp}\, \underline{\mathrm{v}}'xy \wedge S = \sum x \in A(\underline{\mathrm{u}}x \bullet \sum y \in B\underline{\mathrm{v}}'xy) \in \mathrm{rl}$
$\rightarrow S = \sum y \in B \sum x \in A(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy))$

For some notations and some limit theory see Kenyon and Morse.

1.35 $(\bigwedge x \in A \bigwedge y \in B\, \mathrm{rilp}\, \underline{\mathrm{u}}'xy \rightarrow \sum x \in A \sum y \in B\underline{\mathrm{u}}'xy = \sum y \in B \sum x \in A\underline{\mathrm{u}}'xy)$

An easy consequence of either 1.32 or 1.34 is
1.36 $(\sum x(\underline{\mathrm{u}}x \bullet \sum y \in B\, \mathrm{Cr}\, x\underline{\mathrm{v}}y) \in \mathrm{rl}$
$\rightarrow \sum x(\underline{\mathrm{u}}x \bullet \sum y \in B\, \mathrm{Cr}\, x\underline{\mathrm{v}}y) = \sum y \in B \sum x \in \underline{\mathrm{v}}y\underline{\mathrm{u}}x)$

Summation by partition

1.37 $(V = \bigvee y \in B\underline{\mathrm{v}}y \wedge \bigwedge y, z, \in B(y \neq z \to \underline{\mathrm{v}}y \cap \underline{\mathrm{v}}z = 0) \wedge \sum x \in V\underline{\mathrm{u}}x \in \mathrm{rl}$
$\to \sum x \in V\underline{\mathrm{u}}x = \sum y \in B \sum x \in \underline{\mathrm{v}}y\underline{\mathrm{u}}x)$

Proof:

Note that

$(\mathrm{Cr}\, xV = \sum y \in B\, \mathrm{Cr}\, x\underline{\mathrm{v}}y)$

and apply 1.36.

1.38 $(V = \bigvee y \in B\underline{\mathrm{v}}y \wedge \bigwedge x \in V(\underline{\mathrm{u}}x \geq 0) \to 0 \leq \sum x \in V\underline{\mathrm{u}}x \leq \sum y \in B \sum x \in \underline{\mathrm{v}}y\underline{\mathrm{u}}x)$

Proof:

Check that

$(0 \leq \underline{\mathrm{u}}x \bullet \mathrm{Cr}\, xV \leq \underline{\mathrm{u}}x \bullet \sum y \in B\, \mathrm{Cr}\, x\underline{\mathrm{v}}y)$

and apply 1.14 and 1.36.

A fairly evident consequence of 1.36 is

Positive Summation by Partition

1.39 $(V = \bigvee y \in B\underline{\mathrm{v}}y \wedge \bigwedge x \in V\, \mathrm{rilp}\, \underline{\mathrm{u}}x \wedge \bigwedge y \in B \bigwedge z \in B(y \neq z \to \underline{\mathrm{v}}y \cap \underline{\mathrm{v}}z = 0)$
$\to \sum x \in V\underline{\mathrm{u}}x = \sum y \in B \sum x \in \underline{\mathrm{v}}y\underline{\mathrm{u}}x)$

Summation by Transplantation

1.40 $(\text{univalent is } f \to \sum t \in \mathrm{dmn}\, f\underline{\mathrm{u}}\,\boldsymbol{.}\, ft = \sum x \in \mathrm{rng}\, f\underline{\mathrm{u}}x)$

Summation by Commutation

1.41 $(\sum x \in A \sum y \in B|\underline{\mathrm{u}}'xy| < \infty \to \sum x \in A \sum y \in B\underline{\mathrm{u}}'xy = \sum y \in B \sum x \in A\underline{\mathrm{u}}'xy \in \mathrm{rf})$

Proof:

$(\mathrm{rf} \ni \sum x \in A \sum y \in B\, \mathrm{ps}\, \underline{\mathrm{u}}'xy - \sum x \in A \sum y \in B\, \mathrm{ng}\, \underline{\mathrm{u}}'xy$
$= \sum x \in A(\sum y \in B\, \mathrm{ps}\, \underline{\mathrm{u}}'xy - \sum y \in B\, \mathrm{ng}\, \underline{\mathrm{u}}'xy)$
$= \sum x \in A \sum y \in B\underline{\mathrm{u}}'xy$
$= \sum y \in B \sum x \in A\, \mathrm{ps}\, \underline{\mathrm{u}}'xy - \sum y \in B \sum x \in A\, \mathrm{ng}\, \underline{\mathrm{u}}'xy$
$= \sum y \in B(\sum x \in A\, \mathrm{ps}\, \underline{\mathrm{u}}'xy - \sum x \in A\, \mathrm{ng}\, \underline{\mathrm{u}}'xy)$
$= \sum y \in B \sum x \in A\underline{\mathrm{u}}'xy)$

Dominated Summation by Distribution

1.42 $(\sum x \in A(|\underline{\mathrm{u}}x| \bullet \sum y \in B|\underline{\mathrm{v}}'xy|) < \infty \wedge S = \sum x \in A(\underline{\mathrm{u}}x \bullet \sum y \in B\underline{\mathrm{v}}'xy)$
$\to \sum x \in A \sum y \in B(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy) = S = \sum y \in B \sum x \in A(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy))$

1.43 Theorems

.0 $(\mathrm{Sup}\, \omega = \infty)$

.1 $(0 \neq A \subset \omega \to \mathrm{Inf}\, A \in \omega)$

Definition

1.44 $(\mathrm{pwr}'\, A \equiv \sum x \in A1)$

Theorem

1.45 $(\mathrm{pwr}'\, A \in \omega \vee \mathrm{pwr}'\, A = \infty)$

1.46 Theorems

.0 $(0 = \mathrm{pwr}'\, 0 \leq \mathrm{pwr}'\, A)$

.1 $(\mathrm{pwr}'(A \cup B) + \mathrm{pwr}'(A \cap B) = \mathrm{pwr}'\, A + \mathrm{pwr}'\, B)$

.2 $(n \in \omega \to \mathrm{pwr}'\, n = n)$

.3 $(\mathrm{pwr}'\, A = n \in \omega \leftrightarrow \bigvee f(\text{univalent is } f \wedge \mathrm{dmn}\, f = n \wedge \mathrm{rng}\, f = A))$

.4 $(\mathrm{pwr}'\, A \in \omega \leftrightarrow A \in \mathrm{fnt})$

.5 $(\mathrm{pwr}'\, A = \infty \leftrightarrow A \backsim\in \mathrm{fnt})$

Definition

1.47 $(\mathrm{nt}\, ab \equiv \mathsf{E}\, t(a \leq t \leq b \vee b \leq t \leq a))$

Definition

1.48 $(\sum x \in A \underline{\mathrm{u}} x \equiv \mathrm{lin}\, n \sum x \in A \cap \mathrm{nt} - nn\underline{\mathrm{u}}x)$

Note that $(\sum j \in \omega' = 0)$ because of the symmetry employed in 1.48.

1.49 Theorems

.0 $(\sum j \in \omega \underline{\mathrm{u}} j = \mathrm{lin}\, n \sum j \in n \underline{\mathrm{u}} n)$

.1 $(\sum n \in \omega \underline{\mathrm{u}} n \in \mathrm{rl} \to \sum n \in \omega \underline{\mathrm{u}} n = \sum n \in \omega \underline{\mathrm{u}} n)$

.2 $(\sum n \in \omega |\underline{\mathrm{u}} n| = \sum n \in \omega |\underline{\mathrm{u}} n|)$

.3 $(\sum n \in \omega |\underline{\mathrm{u}} n| < \infty \to \sum n \in \omega \underline{\mathrm{u}} n = \sum n \in \omega \underline{\mathrm{u}} n)$

1.50 Exercises

.0 $(0 \leq a \wedge n \in \omega \to {}^{\bullet}(1 + a)n \geq 1 + n \cdot a)$

Hint: induction

.1 $(1 < y \to \mathrm{lin}\, n^{\bullet} yn = \infty)$

.2 $(|x| < 1 \to 0 = \mathrm{lin}\, n^{\bullet}|x|n = \mathrm{lin}\, n^{\bullet} xn)$

.3 $(|x| < \infty \wedge n \in \omega \to (1 - x) \cdot \sum j \in n^{\bullet} xj = 1 - {}^{\bullet} xn)$

.4 $(|x| < 1 \to \sum j \in \omega^{\bullet} xj = 1/(1 - x))$

Chapter 2: Measure

Preliminaries

2.0 Definitions

.0 $(\mathrm{gauge} \equiv \mathrm{To}\,\mathsf{E}\,x(0 \le x \le \infty))$
.1 $(\mathrm{disjointed\,is}\,H \equiv \bigwedge \alpha, \beta, \in H(\alpha \ne \beta \to \alpha\beta = 0))$
.2 $(\mathrm{dsjn} \equiv \mathsf{E}\,H\,\mathrm{disjointed\,is}\,H)$
.3 $({\sim}'F \equiv \bigvee \alpha \in F\,\mathrm{sng}(\nabla F \backsim \alpha))$
.4 $(\nabla'' F \equiv \bigvee G \in \mathrm{cbl} \cap \mathrm{sb}\,F\,\mathrm{sng}\,\nabla G)$
.5 $(\sqcap'' F \equiv \bigvee G \in \mathrm{cbl} \cap \mathrm{sb}\,F\,\mathrm{sng}(\nabla F \sqcap G))$
.6 $(\nabla\mathrm{field} \equiv \mathsf{E}\,F(\nabla'' F = {\sim}'F = F))$
.7 $(\mathrm{Borel}\,F \equiv (\mathrm{sb}\,\nabla F \sqcap \mathsf{E}\,G \in \nabla\mathrm{field}(F \subset G \subset \mathrm{sb}\,\nabla F)))$
.8 $(\mathrm{tract}\,F \equiv \bigvee \alpha \in \nabla'' F\,\mathrm{sb}\,\alpha)$
.9 $(\mathrm{dmn}'\,\varphi \equiv \mathsf{E}\,x(|\,.\,\varphi x| < \infty))$
.10 $(\mathrm{dmn}''\,\varphi \equiv (\mathrm{dmn}\,\varphi \cap \mathrm{tract}\,\mathrm{dmn}'\,\varphi))$
.11 $(\mathrm{rlm}\,\varphi \equiv \nabla\,\mathrm{dmn}\,\varphi)$
.12 $(\mathrm{sprd}\,\varphi \equiv \nabla\,\mathrm{rng}\,\varphi)$
.13 $(\mathrm{zr}\,f \equiv \mathsf{E}\,x(.\,fx = 0))$
.14 $(\mathrm{sct}\,\varphi T \equiv \curlywedge A \in \mathrm{dmn}\,\varphi\,.\,\varphi(TA))$
.15 $(\mathrm{sqnc}{\subset}\,F \equiv \mathsf{E}\,K \in \mathrm{sqnc}\,F \bigwedge n \in \omega(.\,Kn \subset .\,K(n+1)))$
.16 $(\mathrm{sqnc}{\supset}\,F \equiv \mathsf{E}\,K \in \mathrm{sqnc}\,F \bigwedge n \in \omega(.\,Kn \supset .\,K(n+1)))$
.17 $(\mathrm{cover}\,A \equiv \mathsf{E}\,H(A \subset \nabla H))$
.18 $(\mathrm{cuv}\,AH \equiv (\mathrm{cbl} \cap \mathrm{cover}\,A \cap \mathrm{sb}\,H))$

Fundamentals

2.1 Definitions

.0 $(\mathrm{Msr}\,S \equiv \mathsf{E}\,\varphi \in \mathrm{gauge} \cap \mathrm{On}\,\mathrm{sb}\,S \bigwedge A \in \mathrm{dmn}\,\varphi \bigwedge F \in \mathrm{cuv}\,A\,\mathrm{sb}\,S(.\,\varphi A \le \sum \beta \in F\,.\,\varphi\beta))$
.1 $(\mathrm{measure\,is}\,\varphi \equiv (\varphi \in \mathrm{Msr}\,\mathrm{rlm}\,\varphi))$
.2 $(\mathrm{Ms} \equiv \mathsf{E}\,\varphi\,\mathrm{measure\,is}\,\varphi)$
.3 $(\mathrm{measurable}\,\varphi\,\mathrm{is}\,A \equiv (\varphi \in \mathrm{Ms} \wedge A \in \mathrm{dmn}\,\varphi \wedge \bigwedge T \in \mathrm{dmn}\,\varphi(.\,\varphi T = .\,\varphi(TA) + .\,\varphi(T \backsim A))))$
.4 $(\mathrm{mbl}\,\varphi \equiv \mathsf{E}\,A\,\mathrm{measurable}\,\varphi\,\mathrm{is}\,A)$
.5 $(\mathrm{mbl}'\,\varphi \equiv (\mathrm{mbl}\,\varphi \cap \mathrm{dmn}'\,\varphi))$
.6 $(\mathrm{mbl}''\,\varphi \equiv (\mathrm{mbl}\,\varphi \cap \nabla''\,\mathrm{mbl}'\,\varphi))$
.7 $(\mathrm{smsr}\,\varphi \equiv \mathsf{E}\,\psi \bigvee T(\psi = \mathrm{sct}\,\varphi T \wedge \varphi \in \mathrm{Ms}))$
.8 $(\mathrm{submeasure}\,\varphi \equiv \mathrm{smsr}\,\varphi)$
.9 $(\mathrm{sms}\,\varphi \equiv \mathsf{E}\,\psi \in \mathrm{smsr}\,\varphi(.\,\psi\,\mathrm{rlm}\,\psi < \infty))$
.10 $(\mathrm{hull}\,\varphi A \equiv \mathsf{E}\,A'(A \subset A' \in \mathrm{mbl}\,\varphi \wedge .\,\varphi A' = .\,\varphi A))$
.11 $(\mathrm{Hull}\,H \equiv \mathsf{E}\,\varphi \in \mathrm{Msr}\,\nabla H \bigwedge A \in \mathrm{dmn}\,\varphi \bigvee A' \in H \cap \mathrm{sp}\,A(.\,\varphi A' = .\,\varphi A))$
.12 $(\mathrm{Msh} \equiv \mathsf{E}\,\varphi \in \mathrm{Ms}(\varphi \in \mathrm{Hull}\,\mathrm{mbl}\,\varphi))$

Theorems

2.2 $(\varphi \in \mathrm{Msr}\,S \to S = \mathrm{rlm}\,\varphi \wedge \varphi \in \mathrm{Ms})$
2.3 $(\varphi \in \mathrm{Msr}\,S \to .\,\varphi 0 = 0 \wedge (A \subset B \subset S \to .\,\varphi A \le .\,\varphi B) \wedge$
$(H \in \mathrm{cbl} \wedge \nabla H \subset S \to .\,\varphi\,\nabla H \le \sum \beta \in H\,.\,\varphi\beta))$
2.4 $(\varphi \in \mathrm{Ms} \wedge A \cup B \in \mathrm{dmn}\,\varphi \to .\,\varphi(A \cup B) \le .\,\varphi A + .\,\varphi B)$
Hint: let $(F = \mathrm{sng}\,A \cup \mathrm{sng}\,B)$ and use 1.21.1.
2.5 $(A \in \mathrm{mbl}\,\varphi \leftrightarrow \varphi \in \mathrm{Ms} \wedge A \in \mathrm{dmn}\,\varphi \wedge \bigwedge T \in \mathrm{dmn}'\,\varphi \backsim \mathrm{zr}\,\varphi(.\,\varphi T = .\,\varphi(TA) + .\,\varphi(T{\sim}A)))$

2.6 $(\varphi \in \mathrm{Ms} \wedge K \in \mathrm{cbl} \wedge \bigwedge x \in K(\underline{\mathrm{u}}x \in \mathrm{dmn}\,\varphi) \rightarrow .\,\varphi \bigvee x \in K\underline{\mathrm{u}}x \leq \sum x \in K\,.\,\varphi\underline{\mathrm{u}}x)$

Proof:

Because of 1.38

$(H = \bigvee x \in K\,\mathrm{sng}\,\underline{\mathrm{u}}x \rightarrow$

$.\,\varphi \bigvee x \in K\underline{\mathrm{u}}x = .\,\varphi \triangledown H$

$\leq \sum \beta \in H\,.\,\varphi\beta$

$\leq \sum x \in K \sum \beta \in \mathrm{sng}\,\underline{\mathrm{u}}x\,.\,\varphi\beta$

$= \sum x \in K\,.\,\varphi\underline{\mathrm{u}}x)$

2.7 $(\varphi \in \mathrm{Msr}\,S \rightarrow \mathrm{sms}\,\varphi \subset \mathrm{smsr}\,\varphi \subset \mathrm{Msr}\,S)$

Proof:

Evidently $(\mathrm{sms}\,\varphi \subset \mathrm{smsr}\,\varphi)$.

Now suppose $(\psi \in \mathrm{smsr}\,\varphi)$ and choose T so that $(\psi = \mathrm{sct}\,\varphi T)$.

Clearly,

$(\mathrm{sb}\,S = \mathrm{dmn}\,\varphi = \mathrm{dmn}\,\psi \wedge \psi \in \mathrm{gauge} \cap \mathrm{On}\,\mathrm{sb}\,S)$.

Moreover

$(H \in \mathrm{cuv}\,A\,\mathrm{sb}\,S \rightarrow TA \subset T \triangledown H = \bigvee \beta \in H(T\beta) \wedge$

$.\,\psi A = .\,\varphi(TA)$

$\leq .\,\varphi \bigvee \beta \in H\,.\,\varphi(T\beta)$

$\leq \sum \beta \in H\,.\,\varphi(T\beta)$

$= \sum \beta \in H\,.\,\psi\beta)$

Consequently $(\psi \in \mathrm{Msr}\,S)$.

The arbitrary nature of ψ assures us $(\mathrm{smsr}\,\varphi \subset \mathrm{Msr}\,S)$.

2.8 Theorems

.0 $(\psi \in \mathrm{smsr}\,\varphi \rightarrow \bigwedge A \subset \mathrm{rlm}\,\varphi(.\,\psi A \leq .\,\varphi A))$

.1 $(\varphi \in \mathrm{Msr}\,S \rightarrow A \in \mathrm{mbl}\,\varphi \leftrightarrow \bigwedge \psi \in \mathrm{smsr}\,\varphi(.\,\psi S = .\,\psi A + .\,\psi(S{\sim}A)))$

.2 $(\varphi \in \mathrm{Msr}\,S \rightarrow A \in \mathrm{mbl}\,\varphi \leftrightarrow \bigwedge \psi \in \mathrm{sms}\,\varphi(.\,\psi A + .\,\psi(S{\sim}A) \leq .\,\psi S))$

.3 $(\varphi_2 \in \mathrm{smsr}\,\varphi_1 \wedge \varphi_1 \in \mathrm{smsr}\,\varphi \rightarrow \varphi_2 \in \mathrm{smsr}\,\varphi)$

.4 $(\varphi_2 \in \mathrm{sms}\,\varphi_1 \wedge \varphi_1 \in \mathrm{smsr}\,\varphi \rightarrow \varphi_2 \in \mathrm{sms}\,\varphi)$

.5 $(\varphi_1 \in \mathrm{smsr}\,\varphi \rightarrow \varphi_1 \in \mathrm{Msr}\,\mathrm{rlm}\,\varphi \subset \mathrm{Ms})$

.6 $(\psi \in \mathrm{smsr}\,\varphi \rightarrow \mathrm{mbl}\,\varphi \subset \mathrm{mbl}\,\psi)$

.7 $(\psi \in \mathrm{smsr}\,\varphi \rightarrow \mathrm{sct}\,\psi A \in \mathrm{smsr}\,\mathrm{sct}\,\varphi A)$

.8 $(\psi \in \mathrm{sms}\,\varphi \rightarrow \mathrm{sct}\,\psi A \in \mathrm{sms}\,\mathrm{sct}\,\varphi A)$

.9 $(\varphi \in \mathrm{Msr}\,S \wedge S' \subset S \wedge S{\sim}S' \in \mathrm{zr}\,\varphi \rightarrow .\,\varphi(SA) = .\,\varphi(S'A))$

.10 $(\varphi \in \mathrm{Msr}\,S \wedge S' \subset S \wedge \varphi' \in \mathrm{smsr}\,\mathrm{sct}\,\varphi S' \rightarrow .\,\varphi'(SA) = .\,\varphi'(S'A))$

Theorem

2.9 $(A \in \mathrm{mbl}\,\varphi \rightarrow \mathrm{rlm}\,\varphi{\sim}A \in \mathrm{mbl}\,\varphi)$

Proof:

$(\psi \in \mathrm{sms}\,\varphi \wedge S = \mathrm{rlm}\,\varphi \rightarrow$

$.\,\psi S \geq .\,\psi A + .\,\psi(S{\sim}A)$

$= .\,\psi(S{\sim}A) + .\,\psi A$

$= .\,\psi(S{\sim}A) + .\,\psi(S{\sim}(S{\sim}A)))$

Thus because of 2.8.2, $(S{\sim}A \in \mathrm{mbl}\,\varphi)$.

2.10 Theorems

.0 $(\varphi \in \mathrm{Ms} \wedge .\,\varphi A = 0 \rightarrow A \in \mathrm{mbl}\,\varphi)$

.1 $(\varphi \in \mathrm{Ms} \rightarrow 0 \in \mathrm{mbl}\,\varphi \wedge \mathrm{rlm}\,\varphi \in \mathrm{mbl}\,\varphi)$

2.11 Theorems

.0 $(\varphi \in \text{Ms} \wedge T \in \text{dmn}\,\varphi \rightarrow T \in \text{mbl}\,\text{sct}\,\varphi T)$

.1 $(B \in \text{mbl}\,\text{sct}\,\varphi T \wedge T \in \text{mbl}\,\varphi \rightarrow TB \in \text{mbl}\,\varphi)$

Proof:

Let

$(\psi = \text{sct}\,\varphi T)$.

Now suppose

$(S \in \text{dmn}\,\varphi)$.

Since

$(T \in \text{mbl}\,\varphi)$

we know

$(\mathbf{.}\varphi(S{\sim}(TB)) = \mathbf{.}\varphi(S{\sim}(TB)T) + \mathbf{.}\varphi(S{\sim}(TB){\sim}T) = \mathbf{.}\varphi(TS{\sim}B) + \mathbf{.}\varphi(S{\sim}T))$.

Since

$(B \in \text{mbl}\,\psi)$

$(\mathbf{.}\varphi(STB) + \mathbf{.}\varphi(TS{\sim}B) = \mathbf{.}\psi(SB) + \mathbf{.}\psi(S{\sim}B) = \mathbf{.}\psi S = \mathbf{.}\psi(ST))$

Since

$(T \in \text{mbl}\,\varphi)$

we also know

$(\mathbf{.}\varphi(ST) + \mathbf{.}\varphi(S{\sim}T) = \mathbf{.}\varphi S \wedge$

$\mathbf{.}\varphi(STB) + \mathbf{.}\varphi(S{\sim}(TB))$

$= \mathbf{.}\varphi(STB) + \mathbf{.}\varphi(TS{\sim}B) + \mathbf{.}\varphi(S{\sim}T)$

$= \mathbf{.}\varphi(ST) + \mathbf{.}\varphi(S{\sim}T)$

$= \mathbf{.}\varphi S)$.

The arbitrary nature of S assures us $(TB \in \text{mbl}\,\varphi)$.

Theorems

2.12 $(A \in \text{mbl}\,\varphi \wedge B \in \text{mbl}\,\varphi \rightarrow AB \in \text{mbl}\,\varphi)$

Proof:

According to 2.86 $(B \in \text{mbl}\,\text{sct}\,\varphi A)$ and according to 2.11.1 $(AB \in \text{mbl}\,\varphi)$.

2.13 $(A \in \text{mbl}\,\varphi \wedge B \in \text{mbl}\,\varphi \rightarrow A{\sim}B \in \text{mbl}\,\varphi)$

Proof:

$(S = \text{rlm}\,\varphi \rightarrow A{\sim}B = A(S{\sim}B))$

2.14 $(A \in \text{mbl}\,\varphi \wedge B \in \text{mbl}\,\varphi \rightarrow A \cup B \in \text{mbl}\,\varphi)$

Proof:

$(S = \text{rlm}\,\varphi \rightarrow A \cup B = S{\sim}(S{\sim}(A \cup B))$

$= S{\sim}(S{\sim}A{\sim}B) = S{\sim}((S{\sim}A){\sim}B))$

Lemma

2.15 $(\varphi \in \text{Ms} \wedge H \in \text{fnt}\,\text{sb}\,\text{mbl}\,\varphi \rightarrow \nabla H \in \text{mbl}\,\varphi)$

Proof:

Use 2.10.1, 2.14, and induction.

Theorem

2.16 $(A \in \text{mbl}\,\varphi \wedge B \in \text{dmn}\,\varphi \rightarrow \mathbf{.}\varphi(A \cup B) + \mathbf{.}\varphi(AB) = \mathbf{.}\varphi A + \mathbf{.}\varphi B)$

Proof:

Since

$(A \in \mathrm{mbl}\,\varphi)$

we know

$(\boldsymbol{.}\,\varphi(A \cup B) = \boldsymbol{.}\,\varphi((A \cup B)A) + \boldsymbol{.}\,\varphi((A \cup B){\sim}A)$
$= \boldsymbol{.}\,\varphi A + \boldsymbol{.}\,\varphi(B{\sim}A))$.

Also since

$(A \in \mathrm{mbl}\,\varphi)$

we know

$(\boldsymbol{.}\,\varphi(A \cup B) + \boldsymbol{.}\,\varphi(AB) = \boldsymbol{.}\,\varphi A + \boldsymbol{.}\,\varphi(B{\sim}A) + \boldsymbol{.}\,\varphi(AB)$
$= \boldsymbol{.}\,\varphi A + \boldsymbol{.}\,\varphi(BA) + \boldsymbol{.}\,\varphi(B{\sim}A)$
$= \boldsymbol{.}\,\varphi A + \boldsymbol{.}\,\varphi B)$.

Lemma

$(\varphi \in \mathrm{Ms} \wedge H \in \mathrm{fnt\,dsjn\,sb\,mbl}\,\varphi \rightarrow \boldsymbol{.}\,\varphi \triangledown H = \sum \beta \in H \,\boldsymbol{.}\, \varphi\beta)$

Proof:

Use 2.16 and induction.

Theorems

2.17 $(\varphi \in \mathrm{Ms} \wedge H \in \mathrm{cbl\,dsjn\,sb\,mbl}\,\varphi \rightarrow \boldsymbol{.}\,\varphi \triangledown H = \sum \beta \in H \,\boldsymbol{.}\, \varphi\beta)$

Proof:

$(F \in \mathrm{fnt\,sb}\,H \rightarrow \sum \beta \in F \,\boldsymbol{.}\, \varphi\beta = \boldsymbol{.}\,\varphi \triangledown F \leq \boldsymbol{.}\,\varphi \triangledown H \leq \sum \beta \in H \,\boldsymbol{.}\, \varphi\beta)$

Now use 1.2.5.

2.18 $(\boldsymbol{.}\,\varphi 0 = 0 \wedge \bigwedge x \in K(\underline{\mathrm{u}}x \in \mathrm{U}) \wedge \bigwedge x, y, \in K(x \neq y \rightarrow \underline{\mathrm{u}}x \cap \underline{\mathrm{u}}y = 0) \wedge G = \bigvee x \in K\,\mathrm{sng}\,\underline{\mathrm{u}}x$
$\rightarrow \mathrm{disjointed\,is}\,G \wedge \sum x \in K \,\boldsymbol{.}\, \varphi\underline{\mathrm{u}}x = \sum \beta \in G \,\boldsymbol{.}\, \varphi\beta)$

Proof:

Let

$(K' = K \sqsubseteq x(\underline{\mathrm{u}}x \neq 0) \wedge G' = \bigvee x \in K'\,\mathrm{sng}\,x \wedge N = \lambda x \in K'\underline{\mathrm{u}}x)$.

Now

$(x, y, \in K' \wedge x \neq y \rightarrow \underline{\mathrm{u}}x \cap \underline{\mathrm{u}}y = 0$
$\rightarrow \underline{\mathrm{u}}x \neq \underline{\mathrm{u}}y$
$\rightarrow \boldsymbol{.}\,Nx \neq \boldsymbol{.}\,Ny)$.

Hence

$(\mathrm{univalent\,is}\,N \wedge \mathrm{dmn}\,N = K' \wedge \mathrm{rng}\,N = G')$

and because of summation by transplantation it follows that

$(\sum x \in K \,\boldsymbol{.}\, \varphi\underline{\mathrm{u}}x = \sum x \in K' \,\boldsymbol{.}\, \varphi\underline{\mathrm{u}}x + 0$
$= \sum x \in K' \,\boldsymbol{.}\, Nx + 0$
$= \sum \beta \in G' \,\boldsymbol{.}\, \varphi\beta + 0$
$= \sum \beta \in G \,\boldsymbol{.}\, \varphi\beta)$.

2.19 $(\bigwedge H \in \mathrm{cbl} \cap \mathrm{dsjn} \cap \mathrm{sb}\,M(\boldsymbol{.}\,\psi \triangledown H = \sum \beta \in H \,\boldsymbol{.}\, \psi\beta) \wedge K \in \mathrm{cbl} \wedge \bigwedge x \in K(\underline{\mathrm{u}}x \in M)$
$\wedge \bigwedge x, y, \in K(x \neq y \rightarrow \underline{\mathrm{u}}x \cap \underline{\mathrm{u}}y = 0)$
$\rightarrow \boldsymbol{.}\,\psi \bigvee x \in K\underline{\mathrm{u}}x = \sum x \in K \,\boldsymbol{.}\, \psi\underline{\mathrm{u}}x)$

Proof:

Let

$(G = \bigvee x \in K \operatorname{sng} \underline{\mathrm{u}}x)$

and note that

$(G \in \mathrm{cbl} \cap \mathrm{dsjn} \cap \mathrm{sb}\, M \wedge \nabla G = \bigvee x \in K \underline{\mathrm{u}}x \wedge .\, \psi 0 = 0)$

and use 2.18 in checking

$(.\, \psi \bigvee x \in K \underline{\mathrm{u}}x = .\, \psi \nabla G = \sum \beta \in G \,.\, \psi\beta = \sum x \in K \,.\, \psi \underline{\mathrm{u}}x)$.

From 2.17 and 2.18 we have at once

2.20 $(\varphi \in \mathrm{Ms} \wedge K \in \mathrm{cbl} \wedge \bigwedge x \in K(\underline{\mathrm{u}}x \in \mathrm{mbl}\, \varphi) \wedge \bigwedge x, y, \in K(x \neq y \to \underline{\mathrm{u}}x \cap \underline{\mathrm{u}}y = 0)$

$\to .\, \varphi \bigvee x \in K \underline{\mathrm{u}}x = \sum x \in K \,.\, \varphi \underline{\mathrm{u}}x)$

2.21 $(\bigwedge \alpha \in M \bigwedge \beta \in M \cap \mathrm{sb}\, \alpha(\alpha {\sim} \beta \in M) \wedge \bigwedge H \in \mathrm{cbl} \cap \mathrm{dsjn} \cap \mathrm{sb}\, M(.\, \psi \nabla H = \sum \beta \in H \,.\, \psi\beta \in \mathrm{rl})$

$\wedge\, S \in \mathrm{sqnc} \subset M \wedge A = \bigvee n \in \omega \,.\, Sn$

$\to \lim n \,.\, \psi \,.\, Sn = .\, \psi A \in \mathrm{rl})$

Proof:

Let

$(S' = \curlywedge n \in \omega(.\, Sn {\sim} \bigvee \nu \in n \,.\, S\nu) \wedge G = \bigvee n \in \omega \operatorname{sng} .\, S'n)$.

Check that

$(G \in \mathrm{cbl} \cap \mathrm{dsjn} \cap \mathrm{sb}\, M \wedge \bigwedge m \in \omega(.\, Sm = \bigvee \neq\in (n+1) \,.\, S'\nu) \wedge A = \bigvee n \in \omega \,.\, S'n = \nabla G)$

and that 2.19 may be applied twice to yield

$(\mathrm{rl} \ni .\, \psi A = \sum \nu \in \omega \,.\, \psi \,.\, S'\nu$

$= \lim n \sum \nu \in n + 1 \,.\, \psi \,.\, S'\nu$

$= \lim n \,.\, \psi \,.\, Sn)$.

The proof is complete.

From 2.18 and 2.21 we have at once

2.22 $(S \in \mathrm{sqnc} \subset \mathrm{mbl}\, \varphi \wedge A = \bigvee n \in \omega \,.\, Sn \to .\, \varphi A = \lim n \,.\, \varphi \,.\, Sn)$

2.23 $(\varphi \in \mathrm{Ms} \wedge H \in \mathrm{cbl} \cap \mathrm{sb}\, \mathrm{mbl}\, \varphi \to \nabla H \in \mathrm{mbl}\, \varphi)$

Proof:

In view of 2.15 we assume $(H \neq 0)$ and choose $(K \in \mathrm{sqnc}\, H)$ so that $(H = \bigvee n \in \omega \operatorname{sng} .\, Kn)$.

Let

$(S = \mathrm{rlm}\, \varphi \wedge A = \nabla H \wedge K' = \curlywedge n \in \omega \bigvee \nu \in n \,.\, Kn)$.

Note that

$(n \in \omega \to .\, K'n \subset .\, K'(n+1) \subset A \wedge {\sim} .\, K'n \supset {\sim} A)$

and also that

$(A = \bigvee n \in \omega \,.\, K'n)$.

We now assume

$(\psi \in \mathrm{sms}\, \varphi)$.

From 2.15 and 2.8.6 we infer

$(n \in \omega \to .\, \psi S = .\, \psi \,.\, K'n + .\, \psi(S {\sim} .\, K'n) \geq .\, \psi \,.\, S'n + .\, \psi(S {\sim} A))$.

Hence from 2.22 we conclude

$(.\, \psi A + .\, \psi(S {\sim} A) = \lim n \,.\, \psi K'n + .\, \psi(S {\sim} A)$

$= \lim n(.\, \psi \,.\, K'n + .\, \psi(S {\sim} A))$

$\leq .\, \psi S)$.

Because of 2.8.2 the desired conclusion is at hand.

2.24 Theorems

.0 $(F \in \nabla\text{field} \to \prod'' F = F)$

.1 $(F \subset G \wedge \nabla F = \nabla G \to \text{Borel}\, F \subset \text{Borel}\, G)$

.2 $(F \in \text{U} \to \text{Borel}\, F \in \nabla\text{field})$

.3 $(\text{Borel}\,\text{Borel}\, F = \text{Borel}\, F)$

.4 $(F \in \nabla\text{field} \leftrightarrow \text{Borel}\, F = F \in \text{U})$

Theorem

2.25 $(\varphi \in \text{Ms} \to \text{mbl}\,\varphi \in \nabla\text{field})$

2.26 Theorems

.0 $(A \in \text{mbl}''\,\varphi \wedge B \in \text{mbl}''\,\varphi \to AB \in \text{mbl}''\,\varphi)$

.1 $(\varphi \in \text{Ms} \leftrightarrow \nabla''\,\text{mbl}''\,\varphi = \text{mbl}''\,\varphi)$

.2 $(\varphi \in \text{Ms} \wedge F \in \text{cbl} \cap \text{sb}\,\text{mbl}\,\varphi \wedge F\,\text{mbl}''\,\varphi \neq 0 \to \prod F \in \text{mbl}''\,\varphi)$

.3 $(\text{rlm}\,\varphi \in \text{mbl}''\,\varphi \to \text{mbl}\,\varphi = \text{mbl}''\,\varphi)$

Theorems

2.27 $(A \in \text{mbl}'\,\varphi \wedge A \subset B \in \text{dmn}\,\varphi \to .\,\varphi(B{\sim}A) = .\,\varphi B \;—\; .\,\varphi A)$

2.28 $(S \in \text{sqnc} \supset \text{mbl}'\,\varphi \wedge A = \bigvee n \in \omega\,.\,Sn \to .\,\varphi A = \text{lin}\, n\,.\,\varphi\,.\,Sn)$

2.29 $(.\,\varphi A < \infty \wedge T \in \text{mbl}\,\varphi \wedge \psi = \text{sct}\,\varphi T \to \text{hull}\,\varphi A \subset \text{hull}\,\psi A)$

Proof:

Recall 2.8.6 and then check

$$
\begin{aligned}
&(A' \in \text{hull}\,\varphi A \to A \subset A' \in \text{mbl}'\,\varphi \wedge \\
&\quad 0 \le .\,\psi A' \;—\; .\,\psi A \\
&\qquad = .\,\varphi(A'T) \;—\; .\,\varphi(AT) \\
&\qquad \le .\,\varphi(A'T) \;—\; .\,\varphi(AT) + .\,\varphi(A'{\sim}T) \;—\; .\,\varphi(A{\sim}T) \\
&\qquad = .\,\varphi A' \;—\; .\,\varphi A \\
&\qquad = 0 \\
&\to A \subset A' \in \text{mbl}\,\psi \wedge .\,\psi A' = .\,\psi A \\
&\to A' \in \text{hull}\,\varphi A)\ .
\end{aligned}
$$

Exercise

2.30 $(\psi \in \text{Hull}\,\text{mbl}\,\varphi \wedge A' \in \text{dmn}'\,\varphi \cap \text{hull}\,\varphi A \wedge \text{zr}\,\varphi \subset \text{zr}\,\psi \wedge T \in \text{mbl}\,\varphi \to .\,\psi(TA) = .\,\psi(TA'))$

2.31 Theorems

.0 $(T \in H \in \nabla\text{field} \wedge \varphi \in \text{Hull}\, H \to \text{sct}\,\varphi T \in \text{hull}\, H)$

.1 $(T \in \text{mbl}''\,\varphi \wedge H \in \nabla\text{field} \wedge \varphi \in \text{Hull}\, H \to \text{sct}\,\varphi T \in \text{Hull}\, H)$

Hint: use 2.27 and 2.8.9.

Theorems

2.32 $(\varphi \in \text{Msh} \wedge \psi \in \text{smsr}\,\varphi \to \psi \in \text{Msh})$

2.33 $(B_1 \in \text{hull}\,\varphi A_1 \wedge B_2 \in \text{hull}\,\varphi A_2 \to B_1 \cup B_2 \in \text{hull}\,\varphi(A_1 \cup A_2))$

Proof:

Clearly

.0 $(.\,\varphi A_2 = \infty \to B_1 \cup B_2 \in \text{hull}\,\varphi(A_1 \cup A_2))$.

On the other hand helped by 2.27 we infer

$$
\begin{aligned}
(.\varphi A < \infty \wedge T = \operatorname{rlm}\varphi{\sim}B_1 \wedge \psi = \operatorname{sct}\varphi T \rightarrow \\
.\varphi(A_1 \cup A_2) &= .\varphi((A_1 \cup A_2)B_1) + .\varphi((A_1 \cup A_2){\sim}B_1) \\
&= .\varphi(A_1 \cup A_2 B_1) + .\varphi(A_2{\sim}B_1) \\
&\geq .\varphi A_1 + .\psi A_2 \\
&= .\varphi B_1 + .\psi B_2 \\
&= .\varphi B_1 + .\varphi(B_2{\sim}B_1) \\
&= .\varphi(B_1 \cup B_2{\sim}B_1) \\
&= .\varphi(B_1 \cup B_2) \\
&\geq .\varphi(A_1 \cup A_2)) \ .
\end{aligned}
$$

Accordingly

.1 $(.\varphi A_2 < \infty \rightarrow B_1 \cup B_2 \in \operatorname{hull}\varphi(A_1 \cup A_2))$

and the desired conclusion follows from .0 and .1.

Remark. The corresponding conjecture for intersections is false.

2.34 $(\bigwedge n \in \omega(\underline{\mathrm{v}}n \in \operatorname{hull}\varphi\underline{\mathrm{u}}n) \rightarrow \bigvee n \in \omega\underline{\mathrm{v}}n \in \operatorname{hull}\varphi\bigvee n \in \omega\underline{\mathrm{u}}n \wedge$
$\operatorname{lin} m\,.\,\varphi\bigvee n \in m\underline{\mathrm{u}}n = .\,\varphi\bigvee n \in \omega\underline{\mathrm{u}}n)$

Proof:

From 2.33 we infer by induction that

$$(m \in \omega \rightarrow \bigvee n \in m\underline{\mathrm{v}}n \in \operatorname{hull}\varphi\bigvee n \in m\underline{\mathrm{u}}n) \ .$$

Using this and 2.22 we infer

$$
\begin{aligned}
(.\varphi\bigvee n \in \omega\underline{\mathrm{u}}n &\geq \operatorname{lin} m\,.\,\varphi\bigvee n \in m\underline{\mathrm{u}}n \\
&= \operatorname{lin} m\,.\,\varphi\bigvee n \in m\underline{\mathrm{v}}m \\
&= .\varphi\bigvee n \in \omega\underline{\mathrm{v}}n \\
&\geq .\varphi\bigvee n \in \omega\underline{\mathrm{u}}n) \ .
\end{aligned}
$$

2.35 $(T \in \operatorname{mbl}\varphi \wedge H \subset \operatorname{mbl}\varphi \wedge \varphi \in \operatorname{Hull} H \wedge \operatorname{rlm}\varphi \in \operatorname{mbl}''\varphi \wedge \psi = \operatorname{sct}\varphi T$
$\rightarrow H \subset \nabla'' H \subset \operatorname{mbl}\psi \wedge \psi \in \operatorname{Hull}\nabla'' H)$

Because of 2.34 and the principle of choice it is easy to check the very useful

2.36 $(\varphi \in \mathrm{Msh} \wedge S \in \operatorname{sqnc}{\subset}\operatorname{dmn}\varphi \wedge A = \bigvee n \in \omega\,.\,Sn \rightarrow \operatorname{lin} n\,.\,\varphi\,.\,Sn = .\varphi A)$

Remark. Even though $(\varphi \in \mathrm{Msh})$, the restriction in 2.28 that $(S \in \operatorname{sqnc}{\supset}\operatorname{mbl}'\varphi)$ cannot be relaxed in the spirit of 2.36 to the requirement that $(S \in \operatorname{sqnc}{\supset}\operatorname{dmn}'\varphi)$.

2.37 $(\varphi \in \mathrm{Ms} \wedge \bigwedge \alpha \in \operatorname{dmn}\varphi(.\varphi\alpha = \inf \beta \in H \cap \operatorname{sp}\alpha\,.\,\varphi\beta) \wedge A \in \operatorname{dmn}\varphi \wedge$
$\bigwedge T \in H \cap \operatorname{dmn}'\varphi(.\varphi T = .\varphi(TA) + .\varphi(T{\sim}A))$
$\rightarrow A \in \operatorname{mbl}\varphi)$

Proof:

Let

$$(T' \in \operatorname{dmn}'\varphi \wedge \epsilon > 0 \wedge T' \subset T'' \in H \wedge .\varphi T'' < .\varphi T' + \epsilon) \ .$$

Clearly

$$(T'' \in H \cap \operatorname{dmn}'\varphi)$$

and we have

$$
\begin{aligned}
(.\varphi T' + \epsilon > .\varphi T'' \\
&= .\varphi(T''A) + .\varphi(T''{\sim}A) \\
&\geq .\varphi(T'A) + .\varphi(T'{\sim}A) \\
&\geq .\varphi T') \ .
\end{aligned}
$$

The last paragraph assures us

$\bigwedge T \in \mathrm{dmn}'\,\varphi(\mathbf{.}\,\varphi T = \mathbf{.}\,\varphi(TA) + \mathbf{.}\,\varphi(T{\sim}A))$

and reference to 2.25 completes the proof.

We now have at once

2.38 $(\varphi \in \mathrm{Msh} \wedge A \in \mathrm{dmn}\,\varphi \to A \in \mathrm{mbl}\,\varphi \leftrightarrow \bigwedge T \in \mathrm{mbl}'\,\varphi(\mathbf{.}\,\varphi T = \mathbf{.}\,\varphi(TA) + \mathbf{.}\,\varphi(T{\sim}A)))$

2.39 $(\varphi \in \mathrm{Msh} \wedge A \cup B \in \mathrm{dmn}\,\varphi \to \mathbf{.}\,\varphi(A \cup B) + \mathbf{.}\,\varphi(AB) \leq \mathbf{.}\,\varphi A + \mathbf{.}\,\varphi B)$

Proof:

Suppose

$(A \subset A' \in \mathrm{mbl}\,\varphi \wedge \mathbf{.}\,\varphi A = \mathbf{.}\,\varphi A')$

Infer from 2.16

$(.\varphi(A \cup B) + \mathbf{.}\,\varphi(AB) \leq \mathbf{.}\,\varphi(A' \cup B) + \mathbf{.}\,\varphi(A'B)$

$= \mathbf{.}\,\varphi A' + \mathbf{.}\,\varphi B$

$= \mathbf{.}\,\varphi A + \mathbf{.}\,\varphi B)$

2.40 $(\varphi \in \mathrm{Msh} \wedge A \cup B \in \mathrm{mbl}'\,\varphi \wedge \mathbf{.}\,\varphi(A \cup B) = \mathbf{.}\,\varphi A + \mathbf{.}\,\varphi B \to A \in \mathrm{mbl}\,\varphi)$

Proof:

Ascertain

$(A' \in \mathrm{mbl}\,\varphi)$

so that

$(A \subset A' \subset A \cup B \wedge \mathbf{.}\,\varphi A' = \mathbf{.}\,\varphi A)$.

With the proof of 2.39 before us we can deduce

$(\mathbf{.}\,\varphi(A'B) = 0)$.

Since

$(A'{\sim}A = A' \cap (A \cup B) \cap {\sim}A = A'B{\sim}A \subset A'B)$

we learn that

$(\mathbf{.}\,\varphi(A'{\sim}A) = 0 \wedge A'{\sim}A \in \mathrm{mbl}\,\varphi)$.

Since

$(A = A'{\sim}(A'{\sim}A))$

the desired conclusion now follows.

The Measurability of Certain Sets

Deferring proofs until later we now state

2.41 Theorems

.0 $(\psi \in \mathrm{Msr}\,S \wedge \bigwedge n \in \omega(\mathbf{.}\,Kn \subset S) \wedge \mathrm{lin}\,n\,\mathbf{.}\,\psi(S{\sim}\mathbf{.}\,Kn) = 0 \to \mathrm{lin}\,n\,\mathbf{.}\,\psi(\mathbf{.}\,Kn\alpha) = \mathbf{.}\,\psi(S\alpha))$

.1 $(\psi \in \mathrm{Msr}\,S \wedge B \subset S \wedge \mathrm{lin}\,n\,\mathbf{.}\,\psi(S{\sim}\mathbf{.}\,Kn) = 0 \wedge \bigwedge n \in \omega(\mathbf{.}\,\psi(\mathbf{.}\,KnB) + \mathbf{.}\,\psi(\mathbf{.}\,Kn{\sim}B) \leq \mathbf{.}\,\psi S)$
$\to \mathbf{.}\,\psi B + \mathbf{.}\,\psi(S{\sim}B) \leq \mathbf{.}\,\psi S)$

.2 $(\psi \in \mathrm{Msr}\,S \wedge K \in \mathrm{sqnc}{\subset}\,\mathrm{sb}\,S \wedge S{\sim}\bigvee n \in \omega\,\mathbf{.}\,Kn \in \mathrm{zr}\,\psi \wedge \sum n \in \omega\,\mathbf{.}\,\psi(\mathbf{.}\,K(n+1){\sim}\mathbf{.}\,Kn) < \infty$
$\to \mathrm{lin}\,n\,\mathbf{.}\,\psi(S{\sim}\mathbf{.}\,Kn) = 0)$

Proof of .0

$(0 < r < \infty \to$

$\mathrm{big}\,n(\mathbf{.}\,\psi(S\alpha) \geq \mathbf{.}\,\psi(S\,\mathbf{.}\,Kn\alpha)$

$= \mathbf{.}\,\psi(S\alpha\,\mathbf{.}\,Kn) + r - r$

$\geq \mathbf{.}\,\psi(S\alpha\,\mathbf{.}\,Kn) + \mathbf{.}\,\psi(S\alpha{\sim}\mathbf{.}\,Kn) - r$

$\geq \mathbf{.}\,\psi(S\alpha) - r)$

Proof of .1

Notice that $\bigwedge n \in \omega(\mathbf{.}\,Kn \subset S)$ and use .0

Proof of .2

Let

$(B = \bigvee n \in \omega \,.\, Kn)$

and note

$(n \in \omega \to S{\sim}\,.\,Kn \subset S{\sim}B \cup B{\sim}\,.\,Kn \wedge$
$B{\sim}\,.\,Kn \subset \bigvee m \in \omega \backsim n(.\,K(m+1){\sim}\,.\,Km) \wedge$
$.\,\psi(S{\sim}\,.\,Kn) \leq .\,\psi(S{\sim}B) + .\,\psi(B{\sim}\,.\,Kn)$
$= .\,\psi(B{\sim}\,.\,Kn)$
$\leq \sum m \in \omega \backsim n\,.\,\psi(.\,K(m+1){\sim}\,.\,Km))$.

Hence

$(0 \leq \operatorname{lin} n\,.\,\psi(S{\sim}\,.\,Kn)$
$\leq \operatorname{lin} n \sum m \in \omega \backsim n(.\,\psi(.\,K(m+1){\sim}\,.\,Km)$
$= 0)$.

2.42 $(\psi \in \operatorname{Msr} S \wedge A \in \operatorname{sqnc}{\subset}\,\mathrm{U} \wedge$
$\bigwedge n \in \omega(.\,\psi\,.\,An + .\,\psi(.\,A(n+1){\sim}\,.\,An) = .\,\psi\,.\,A(n+2))$
$\to \sum n \in \omega\,.\,\psi(.\,A(n+1){\sim}\,.\,An) \leq 2 \cdot .\,\psi S)$

Proof:

We assume

$(.\,\psi S < \infty)$

since otherwise the desired conclusion is obvious.

Let

$(M = .\,\psi\,.\,A0 + .\,\psi\,.\,A1)$

and complete the proof in four steps, the first of which is obvious and the second of which was suggested by Maurice Sion.

Step 1

$(n \in \omega \to .\,\psi(.\,A(n+2){\sim}\,.\,A(n+1)) \leq .\,\psi\,.\,A(n+2) \;—\; .\,\psi\,.\,An)$

Step 2

$(N \in \omega \to \sum n \in N\,.\,\psi(.\,A(n+2){\sim}\,.\,A(n+1)) \leq .\,\psi\,.\,AN + .\,\psi\,.\,A(N+1) \;—\; M)$

Proof: use induction on N

Step 3

$(\sum n \in \omega\,.\,\psi(.\,A(n+2){\sim}\,.\,A(n+1)) \leq 2 \cdot .\,\psi S \;—\; M)$

Proof:

$(\sum n \in \omega\,.\,\psi(.\,A(n+2){\sim}\,.\,A(n+1))$
$= \operatorname{lin} N \sum n \in N\,.\,\psi(.\,A(n+2){\sim}\,.\,A(n+1))$
$\leq \operatorname{lin} N(.\,\psi\,.\,AN + .\,\psi\,.\,A(N+1) \;—\; M)$
$\leq \operatorname{lin} N(.\,\psi S + .\,\psi S \;—\; M)$
$= 2 \cdot .\,\psi S \;—\; M)$

Step 4

$(\sum n \in \omega\,.\,\psi(.\,A(n+1){\sim}\,.\,An) \leq 2 \cdot .\,\psi S)$

Proof:

$(\sum n \in \omega\,.\,\psi(.\,A(n+1){\sim}\,.\,An)$
$= .\,\psi(.\,A1{\sim}\,.\,A0) + \sum n \in \omega\,.\,\psi(.\,A(n+2){\sim}\,.\,A(n+1))$
$\leq .\,\psi\,.\,A1 + 2 \cdot .\,\psi S \;—\; M$
$= 2 \cdot .\,\psi S \;—\; .\,\psi A0$
$\leq 2 \cdot .\,\psi S)$

2.43 $(\varphi \in \mathrm{Msr}\, S \wedge A \in \mathrm{sqnc}{\subset}\, \mathrm{U} \wedge B = \bigvee n \in \omega\,.\,An\, \wedge$

$\bigwedge \psi \in \mathrm{sms}\,\varphi \bigwedge n \in \omega(.\,\psi\,.\,An + .\,\psi(S{\sim}\,.\,A(n+1)) \le .\,\psi S)$

$\to B \in \mathrm{mbl}\,\varphi)$

Proof:

We note that

$(B \subset S)$.

Let

$(C = S{\sim}B \wedge K = \curlywedge n \in \omega(C \cup .\,An))$

and complete the proof in five steps.

Step 1

$(n \in \omega \wedge \psi \in \mathrm{sms}\,\varphi \to .\,\psi\,.\,An + .\,\psi(.\,A(n+2){\sim}\,.\,A(n+1))) \le .\,\psi\,.\,A(n+2))$

Proof:

Let

$(\psi' = \mathrm{sct}\,\psi\,.\,A(n+2))$.

Note that

$(.\,\psi\,.\,An + .\,\psi(.\,A(n+2){\sim}\,.\,A(n+1)) = .\,\psi'\,.\,An + .\,\psi'(S{\sim}\,.\,A(n+1))$

$\le .\,\psi' S = .\,\psi\,.\,A(n+2))$.

Step 2

$(\psi \in \mathrm{sms}\,\varphi \to \mathrm{lin}\, n\,.\,\psi(B{\sim}\,.\,An) = 0)$

Proof: lineb Use Step 1, 2.4.2, and 2.41.2.

Step 3

$(\psi \in \mathrm{sms}\,\varphi \to \mathrm{lin}\, n\,.\,\psi(S{\sim}\,.\,Kn) = 0)$

Proof:

$(n \in \omega \to S{\sim}\,.\,Kn = (C \cup B){\sim}(C \cup .\,Kn) = B{\sim}\,.\,An)$

Now use Step 2.

Step 4

$(\psi \in \mathrm{sms}\,\varphi \wedge n \in \omega \to .\,\psi(.\,KnB) + .\,\psi(.\,Kn{\sim}B) \le .\,\psi S)$

Proof:

$(.\,\psi(.\,KnB) + .\,\psi(.\,Kn{\sim}B) = .\,\psi((C \cup .\,An)B) + .\,\psi(.\,Kn{\sim}B)$

$= .\,\psi\,.\,An + .\,\psi(.\,Kn{\sim}B)$

$\le .\,\psi\,.\,An + .\,\psi(S{\sim}B)$

$\le .\,\psi\,.\,An + .\,\psi(S{\sim}\,.\,A(n+1))$

$\le .\,\psi S)$

Step 5

$(B \in \mathrm{mbl}\,\varphi)$

Proof:

Use Step 4, Step 3, 2.41.1, and 2.8.2.

The next two theorems are due to Trevor J. McMinn. The second follows easily from the first whose proof we shall give presently.

2.44 Theorems

.0 $(\varphi \in \mathrm{Msr}\, S \wedge K \in \mathrm{sqnc}{\subset}\, F \wedge S{\sim}\bigvee n \in \omega\,.\,Kn \in \mathrm{zr}\,\varphi \wedge \bigwedge n \in \omega(F \subset \mathrm{mbl}\,\mathrm{sct}\,\varphi\,.\,Kn)$

$\to F \subset \mathrm{mbl}\,\varphi)$

.1 $(\varphi \in \mathrm{Msr}\, S \wedge \bigwedge \psi \in \mathrm{sms}\,\varphi \bigvee K \in \mathrm{sqnc}{\subset}\, F($

$S{\sim}\bigvee n \in \omega\,.\,Kn \in \mathrm{zr}\,\psi \wedge \bigwedge n \in \omega(F \subset \mathrm{mbl}\,\mathrm{sct}\,\psi\,.\,Kn))$

$\to F \subset \mathrm{mbl}\,\varphi)$

Proof of .0:
The desired conclusion is a consequence of 2.8.2 and the statement

$(B \in F \wedge \varphi \in \mathrm{sms}\,\varphi \rightarrow .\,\psi B + .\,\psi(S{\sim}B) \leq .\,\varphi S)$

Proof:
Clearly

$(F \subset \mathrm{sb}\,S)$.

Next

$(N \in \omega \wedge \psi' = \mathrm{sct}\,\psi\,.\,KN$
$\rightarrow F \subset \mathrm{mbl}\,\psi'$
$\rightarrow .\,\psi(.\,KNB) + .\,\psi(.\,KN{\sim}B)$
$= .\,\psi'(.\,KNB) + .\,\psi'(.\,KN{\sim}B)$
$= .\,\psi'\,.\,KN = .\,\psi' S \leq .\,\psi S \wedge$
$\sum n \in N\,.\,\psi(.\,K(n+1){\sim}\,.\,Kn) = \sum n \in N\,.\,\psi(.\,KN\,.\,K(n+1){\sim}\,.\,Kn)$
$= \sum n \in N\,.\,\psi'(.\,K(n+1){\sim}\,.\,Kn)$
$= .\,\psi' \bigvee n \in N(.\,K(n+1){\sim}\,.\,Kn)$
$\leq .\,\psi' S$.
$\leq .\,\psi S)$

Accordingly

$(N \in \omega \rightarrow .\,\psi(.\,KNB) + .\,\psi(.\,KN{\sim}B) \leq .\,\psi S \wedge$
$\sum n \in N\,.\,\psi(.\,K(n+1){\sim}\,.\,Kn) \leq .\,\psi S)$.

Clearly now

.2 $\bigwedge n \in \omega(.\,\psi(.\,KnB) + .\,\psi(.\,Kn{\sim}B) \leq .\,\psi S)$

and

.3 $(\sum n \in \omega\,.\,\psi(.\,K(n+1){\sim}\,.\,Kn) \leq .\,\psi S < \infty)$.

Since evidently

$(S{\sim} \bigvee n \in \omega\,.\,Kn \in \mathrm{zr}\,\psi)$

it now follows from .3 and 2.41.2 that

$(\mathrm{lin}\,n\,.\,\psi(S{\sim}\,.\,Kn) = 0)$.

From this, .2, and 2.41.1 follows the conclusion now desired.

Metric Fundamentals

Definition
2.45 $(\mathrm{Md}\,\rho \equiv \text{Ǝ}\, \varphi \in \mathrm{Msr\,space}\,\rho(\rho \in \mathrm{metric} \wedge \bigwedge A \bigwedge B(\mathrm{dist}\,\rho AB > 0 \rightarrow .\,\varphi A + .\,\varphi B = .\,\varphi(A \cup B))))$

Theorems
2.46 $(\varphi \in \mathrm{Md}\,\rho \wedge \psi \in \mathrm{smsr}\,\varphi \rightarrow \psi \in \mathrm{Md}\,\rho)$
2.47 $(\varphi \in \mathrm{Md}\,\rho \wedge B \in \mathrm{open}\,\rho \rightarrow B \in \mathrm{mbl}\,\varphi)$

Proof:
Letting
$(S = \mathrm{space}\,\rho \wedge \delta = \measuredangle x \in S \inf z \in S \backsim B\,.\,\rho(x,z) \wedge A = \measuredangle n \in \omega\, \text{Ǝ}\, x(.\,\delta x \geq \underline{2}\,n))$
we divide the remainder of the proof into four parts, the first of which is easily checked.
Part 0 $(B = \bigvee n \in \omega\,.\,An)$
Part 1 $(n \in \omega \rightarrow \mathrm{dist}\,\rho\,.\,An(S{\sim}\,.\,A(n+1)) > 0)$
Proof:
Suppose

$(x \in .\,An \wedge y \in S{\sim}\,.\,A(n+1))$.

Since

$(z \in S{\sim}B \rightarrow .\, \delta x \leq .\, \rho(x,z) \leq .\, \rho(x,y) + .\, \rho(y,z))$

we see

$(.\, \delta x \leq .\, \rho(x,y) + .\, \delta y \wedge \underline{2}\, n \leq .\, \delta x \wedge \underline{2}(n+1) > .\, \delta y \wedge$
$\underline{2}\, n \leq .\, \rho(x,y) + \underline{2}(n+1) \wedge 0 < \underline{2}(n+1) = \underline{2}\, n \,—\, \underline{2}(n+1) \leq .\, \rho(x,y))$.

The arbitrary nature of (x,y) assures us that

$(\mathrm{dist}\, \rho\, .\, An(S{\sim}\, .\, A(n+1)) \geq \underline{2}(n+1) > 0)$.

Part 2 $(\psi \in \mathrm{sms}\, \varphi \wedge n \in \omega \rightarrow .\, An \subset .\, A(n+1) \wedge .\, \psi\, .\, An + .\, \psi(S{\sim}\, .\, A(n+1)) \leq .\, \psi S)$

Proof:

Obviously

$(.\, An \subset .\, A(n+1))$.

Moreover, 2.46 assures us

$(\psi \in \mathrm{Md}\, \rho)$

and the remainder of the proof follows from Part 1 and 2.45.

Part 3 $(B \in \mathrm{mbl}\, \varphi)$

Proof: Use Part 0, Part 2, and 2.43.

2.48 $(\rho\, \mathrm{metrizes}\, S \wedge \varphi \in \mathrm{Msr}\, S \rightarrow \varphi \in \mathrm{Md}\, \rho \leftrightarrow \mathrm{open}\, \rho \subset \mathrm{mbl}\, \varphi \leftrightarrow \mathrm{closed}\, \rho \subset \mathrm{mbl}\, \varphi)$

2.49 $(\rho\, \mathrm{metrizes}\, S \wedge \varphi \in \mathrm{Msr}\, S \wedge \bigwedge A, B, \in \mathrm{bounded}\, \rho(\mathrm{dist}\, \rho AB > 0 \rightarrow .\, \varphi A + .\, \varphi B = .\, \varphi(A \cup B))$
$\rightarrow \varphi \in \mathrm{Md}\, \rho)$

Constructed Measures

2.50 Definitions

.0 $(\mathrm{dmtr}\, \rho n \equiv \mathsf{E}\, \beta(\rho \in \mathrm{metric} \wedge \mathrm{diam}\, \rho\beta < \underline{2}\, n))$

.1 $(\mathrm{dsn}'\, H \equiv \bigvee G \in \mathrm{fnt} \cap \mathrm{dsjn} \cap \mathrm{sb}\, H\, \mathrm{sng}\, \nabla\, G)$

.2 $(\mathrm{grated\, is}\, H \equiv \bigwedge \alpha, \beta, \in H(\alpha{\sim}\beta \in \mathrm{dsn}'\, H))$

.3 $((G \subset\subset F) \equiv \bigwedge \alpha \in G \bigvee \beta \in F(\alpha \subset \beta))$

.4 $(\mathrm{refined\, is}\, H \equiv \bigwedge F \in \mathrm{cbl} \cap \mathrm{sb}\, H \bigvee G \in \mathrm{cbl} \cap \mathrm{dsjn} \cap \mathrm{sb}\, H(G \subset\subset F \wedge \nabla\, G = \nabla\, F))$

2.51 Definitions

.0 $(\mathrm{mss}\, gSH \equiv \curlywedge A \subset S \inf F \in \mathrm{cuv}\, AH \sum \beta \in F|\, .\, g\beta|)$

.1 $(\mathrm{msm}\, g\rho H \equiv \curlywedge A \subset \mathrm{space}\, \rho \sup n \in \omega\, .\, \mathrm{mss}\, g\, \mathrm{space}\, \rho(H\, \mathrm{dmtr}\, \rho n)A)$

.2 $(\mathrm{mr}\, g \equiv \mathrm{mss}\, g\, \mathrm{rlm}\, g\, \mathrm{dmn}\, g)$

2.52 Definitions

.0 $(\mathrm{om}\, \varphi \equiv \mathrm{mss}\, \varphi\, \mathrm{rlm}\, \varphi\, \mathrm{mbl}\, \varphi)$

Thus $\mathrm{om}\, \varphi$ is the outer measure associated with φ . For inner measure

.1 $(\mathrm{im}\, \varphi \equiv \curlywedge A \subset \mathrm{rlm}\, \varphi \sup \beta \in \mathrm{sb}\, A \cap \mathrm{mbl}\, \varphi\, .\, \varphi\beta)$

2.53 Definitions

.0 $(\mathrm{add}' \equiv \mathsf{E}\, g \in \mathrm{To\, rl} \bigwedge A \in \mathrm{dmn}\, g \bigwedge F \in \mathrm{fnt} \cap \mathrm{dsjn} \cap \mathrm{sb}\, \mathrm{dmn}\, g(A = \nabla\, F \rightarrow .\, gA = \sum \beta \in F\, .\, g\beta))$

.1 $(\mathrm{add}'' \equiv \mathsf{E}\, g \in \mathrm{To\, rl} \bigwedge A \in \mathrm{dmn}\, g \bigwedge F \in \mathrm{cbl} \cap \mathrm{dsjn} \cap \mathrm{sb}\, \mathrm{dmn}\, g(A = \nabla\, F \rightarrow .\, gA = \sum \beta \in F\, .\, g\beta))$

.2 $(\mathrm{subadd}'' \equiv \mathsf{E}\, g \in \mathrm{gauge} \bigwedge A \in \mathrm{dmn}\, g \bigwedge F \in \mathrm{cbl} \cap \mathrm{dsjn} \cap \mathrm{sb}\, \mathrm{dmn}\, g \cap \mathrm{sb}\, \mathrm{sb}\, A(.\, gA \geq \sum \beta \in F\, .\, g\beta))$

Lemma

2.54 $(F \in \mathrm{cbl} \wedge 0 < \epsilon \rightarrow \bigvee \eta \in \mathrm{To\, rfp}(\sum \beta \in F\, .\, \eta\beta < \epsilon))$

Theorem
2.55 $(S \in \mathrm{U} \wedge \varphi = \mathrm{mss}\, gSH \rightarrow \varphi \in \mathrm{Msr}\, S)$

Proof:

Part 0 $(\varphi \in \mathrm{gauge} \cap \mathrm{On}\, \mathrm{sb}\, S)$

Part 1 $(F \in \mathrm{cuv}\, A\, \mathrm{sb}\, S \rightarrow .\, \varphi A \leq \sum \beta \in F\, .\, \varphi\beta)$

Proof:

We shall assume

.0 $\bigwedge \beta \in F(.\, \varphi\beta < \infty)$

since otherwise Part 0 assures us immediately of the desired result.

Let

$(\epsilon > 0)$

and choose

$(\eta \in \mathrm{To}\, \mathrm{rfp})$

that

$(\sum \beta \in F\, .\, \eta\beta < \epsilon)$.

Relying heavily on .0 and 2.51.0 choose

$(G \in \mathrm{On}\, F)$

so that

$\bigwedge \beta \in F(.\, G\beta \in \mathrm{cuv}\, AH \wedge \sum \alpha \in .\, G\beta |\, .\, g\alpha| \leq .\, \varphi\beta + .\, \eta\beta)$.

Letting

$(F' = \bigvee \beta \in F\, .\, G\beta)$

we see

$$\begin{aligned}
(A \subset \nabla F = \bigvee \beta \in F\beta \subset \bigvee \beta \in F\, \nabla .\, GA = \nabla F' \wedge \\
F' \in \mathrm{cbl} \cap \mathrm{sb}\, H \cap \mathrm{cover}\, A \wedge F' \in \mathrm{cuv}\, AH \wedge \\
.\, \varphi A \leq \sum \alpha \in F' |\, .\, g\alpha| \\
\leq \sum \beta \in F \sum \alpha \in .\, G\beta |\, .\, g\alpha| \\
\leq \sum \beta \in F(.\, \varphi\beta + .\, \eta\beta) \\
= \sum \beta \in F\, .\, \varphi\beta + \sum \beta \in F\, .\, \eta\beta \\
\leq \sum \beta \in F\, .\, \varphi\beta + \epsilon)\ .
\end{aligned}$$

The arbitrary nature of ϵ completes the proof.

Part 2 $(\varphi \in \mathrm{Msr}\, S)$

Proof:

Recall 2.1.0 and use parts 0 and 1.

2.56 Theorems

.0 $(A \subset S \in \mathrm{U} \wedge H \subset H' \rightarrow .\, \mathrm{mss}\, gSH'A \leq .\, \mathrm{mss}\, gSHA)$

.1 $(\varphi \in \mathrm{Msr}\, S \wedge A \subset S \rightarrow .\, \varphi A \leq .\, \mathrm{mss}\, \varphi SHA)$

.2 $(\varphi \in \mathrm{Msr}\, A \rightarrow \varphi = \mathrm{mss}\, \varphi S\, \mathrm{sb}\, S)$

.3 $(A \subset S \in \mathrm{U} \wedge \bigwedge \beta \in H(|\, .\, g\beta| \leq |\, .\, g'\beta|) \rightarrow .\, \mathrm{mss}\, gSHA \leq .\, \mathrm{mss}\, g'SHA)$

Lemma
2.57 $(\inf F \in \mathrm{cuv}\, AH\, .\, g \nabla F = \inf \beta \in \nabla'' H \cap \mathrm{sp}\, A\, .\, g\beta)$

Lemma
2.58 $(\varphi = \mathrm{mss}\, gSH \wedge \beta \in H \wedge \alpha \subset \beta S \rightarrow .\, \varphi\alpha \leq |\, .\, g\beta|)$

Theorems

2.59 $(H = \nabla'' H \wedge \bigwedge F \in \text{cbl} \cap \text{sb}\, H(0 \leq . g \nabla F \leq \sum \beta \in F . g\beta) \wedge S \in \text{U} \wedge \varphi = \text{mss}\, gSH \rightarrow$
$\bigwedge A \subset S(. \varphi A = \inf \beta \in H \cap \text{sp}\, A . g\beta))$

Proof:

Clearly

$$\bigwedge \beta \in H(0 \leq . g\beta) .$$

Hence

$$\begin{aligned}(A \subset S \rightarrow . \varphi A &= \inf F \in \text{cuv}\, AH \textstyle\sum \beta \in F . g\beta \\ &\geq \inf F \in \text{cuv}\, AH . g \nabla F \\ &= \inf \beta \in \nabla'' H \cap \text{sp}\, A . g\beta \\ &= \inf \beta \in H \cap \text{sp}\, A . g\beta \\ &\geq . \varphi A) .\end{aligned}$$

2.60 $(H \subset H' \wedge \nabla H' = S \in \text{U} \wedge \varphi = \text{mss}\, gSH \rightarrow$

.0 $\varphi = \text{mss}\, \varphi SH \in \text{Msr}\, S \wedge$

.1 $\bigwedge A \subset S(. \varphi A = \inf \beta \in \nabla'' H \cap \text{sp}\, A . \varphi\beta) \wedge$

.2 $\varphi \in \text{Hull} \prod'' \nabla'' H')$

Proof:

Since

$$(\varphi \in \text{Msr}\, S)$$

and since

$$\begin{aligned}(A \subset S \rightarrow . \varphi A &= \inf F \in \text{cuv}\, AH \textstyle\sum \beta \in F| . g\beta| \\ &\geq \inf F \in \text{cuv}\, AH \textstyle\sum \beta \in F . \varphi\beta \\ &\geq \inf F \in \text{cuv}\, AH . \varphi \nabla F \\ &= \inf \beta \in \nabla'' H \cap \text{sp}\, A . \varphi\beta \\ &\geq . \varphi A) ,\end{aligned}$$

we easily infer .0 and .1. We deduce .2 from the statement

$$(A \subset S \rightarrow \bigvee A' \in \textstyle\prod'' \nabla'' H' \cap \text{sp}\, A(. \varphi A' = . \varphi A))$$

Proof:

We can use .1 to so ascertain

$$(S' \in \text{sqnc}(\textstyle\prod'' \nabla'' H' \cap \text{sp}\, A)$$

that

$$\bigwedge n \in \omega(. \varphi . S'n \leq . \varphi A + \underline{2}\, n) .$$

Let $(A' = \bigwedge n \in \omega . S'n)$.

Clearly

$$\begin{aligned}(A \subset A' \in \textstyle\prod'' \prod'' \nabla'' H' = \prod'' \nabla'' H' \wedge \\ \bigwedge n \in \omega(. \varphi A' \leq . \varphi . S'n \leq . \varphi A + \underline{2}\, n) \wedge \\ . \varphi A' \leq . \varphi A \leq . \varphi A' \wedge . \varphi A' = . \varphi A)\end{aligned}$$

2.60A $(A \cup B \subset S \in \text{U} \wedge \varphi = \text{mss}\, gSH \wedge \bigwedge \beta \in H(. \varphi(A\beta) + . \varphi(B\beta) \leq | . g\beta|)$
$\rightarrow \bigwedge \psi \in \text{smsr}\, \varphi(. \psi A + . \psi B \leq . \psi S))$

Proof:

Suppose

$$(\psi \in \text{smsr}\, \varphi) ;$$

so select T that

$$(\psi = \text{sct}\, \varphi T) .$$

Suppose

$$(F \in \text{cuv}(ST)H) .$$

Then

$$(\sum\beta\in F|\,.\,g\beta|\geq\sum\beta\in F(.\,\varphi(A\beta)+.\,\varphi(B\beta))$$
$$=\sum\beta\in F\,.\,\varphi(A\beta)+\sum\beta\in F\,.\,\varphi(B\beta)$$
$$\geq .\,\varphi\bigvee\beta\in F(A\beta)+.\,\varphi\bigvee\beta\in F(B\beta)$$
$$=.\,\varphi(A\triangledown F)+.\,\varphi(B\triangledown F)$$
$$\geq .\,\varphi(AST)+.\,\varphi(BST)$$
$$=.\,\varphi(AT)+.\,\varphi(BT))\ .$$

The arbitrary nature of F assures us

$$(.\,\psi S=.\,\varphi(ST)$$
$$=\inf F\in\operatorname{cuv}(ST)H\sum\beta\in F|\,.\,g\beta|$$
$$\geq .\,\varphi(AT)+.\,\varphi(BT)$$
$$=.\,\psi A+.\,\psi B)\ .$$

2.61 $(A\subset S\in \mathrm{U}\wedge\varphi=\operatorname{mss}gSH\wedge$
$\bigwedge T\in H\cap\operatorname{dmn}'g\backsim\operatorname{zr}\varphi(.\,\varphi(AT)+.\,\varphi(ST{\sim}A)\leq|\,.\,gT|)$
$\rightarrow A\in\operatorname{mbl}\varphi)$

Proof:

Note that

$$\bigwedge T\in H(.\,\varphi(TA)+.\,\varphi(S{\sim}AT)\leq|\,.\,gT|)$$

and use 2.60A and 2.8.2.

2.62 $(\triangledown H\subset S\in \mathrm{U}\wedge\varphi=\operatorname{mss}gSH\wedge A\subset S\wedge$
$\bigwedge T\in H\cap\operatorname{dmn}'g\backsim\operatorname{zr}g(.\,\varphi T=.\,\varphi(TA)+.\,\varphi(T{\sim}A))$
$\rightarrow A\in\operatorname{mbl}\varphi)$

2.63 $(S\in\mathrm{U}\wedge g\in\mathrm{gauge}\wedge\psi=\operatorname{mss}gSH\wedge\bigwedge A\in H\bigwedge F\in\operatorname{cuv}AH(.\,gA\leq\sum\beta\in F\,.\,g\beta)\rightarrow$
.0 $(A\subset S\rightarrow .\,\psi A\leq .\,\varphi A)\wedge$
.1 $(A\in\operatorname{mbl}\varphi\wedge B\in H\cap\operatorname{dmn}'\varphi\rightarrow .\,\psi(BA)=.\,\varphi(BA)))$

2.64 $(\triangledown H\subset S\wedge\varphi=\operatorname{mss}gSH\wedge\psi\in\operatorname{Msr}S\wedge\bigwedge\beta\in H(.\,\psi\beta=.\,\varphi\beta)\rightarrow$
.0 $(A\subset S\rightarrow .\,\psi A\leq .\,\varphi A)\wedge$
.1 $(A\in\operatorname{mbl}\varphi\wedge B\in H\cap\operatorname{dmn}'\varphi\rightarrow .\,\psi(BA)=.\,\varphi(BA)))$

After checking .0 we use it in the

Proof of .1

$$(0\leq .\,\varphi(BA)-\ .\,\psi(BA)$$
$$\leq .\,\varphi(BA)-\ .\,\psi(BA)+.\,\varphi(B{\sim}A)-\ .\,\psi(B{\sim}A)$$
$$=.\,\varphi B-\ .\,\psi(BA)-\ .\,\psi(B{\sim}A)$$
$$\leq .\,\varphi B-\ .\,\psi B$$
$$=0)$$

2.65 $(\varphi=\operatorname{mss}gSH\wedge\psi\in\operatorname{Msr}S\wedge\bigwedge\beta\in H(.\,\psi\beta=.\,\varphi\beta)\wedge H\subset\operatorname{mbl}\varphi\operatorname{mbl}\psi\rightarrow$
.0 $\operatorname{mbl}'\varphi\subset\operatorname{mbl}'\psi\wedge$
.1 $\operatorname{mbl}''\varphi\subset\operatorname{mbl}''\psi\wedge$
.2 $\bigwedge\beta\in\operatorname{mbl}''\varphi(.\,\psi\beta=.\,\varphi\beta))$

Proof:

We let

$$(H'=H\cup\operatorname{sng}S)$$

and complete the proof in 5 steps.

Step 0 ($A \in \text{mbl}'\,\varphi \to A \in \text{mbl}'\,\psi$)
Proof:
Choose
$$(A' \in \sqcap''\ \nabla''\ H' \cap \text{sp}\,A)$$
so that
$$(0 \le .\varphi A' = .\varphi A < \infty)\ .$$
Evidently
$$(.\psi(A' \sim A) \le .\varphi(A' \sim A) = 0)$$
and
$$(A' \in \text{mbl}\,\psi \wedge A' \sim A \in \text{mbl}\,\psi \wedge A = A' \sim (A' \sim A) \in \text{mbl}\,\psi \wedge$$
$$0 \le .\psi A \le .\varphi A < \infty \wedge A \in \text{mbl}'\,\psi)\ .$$
The next two steps follow easily.
Step 1 ($\text{mbl}'\,\varphi \subset \text{mbl}'\,\psi$)
Step 2 ($\text{mbl}''\,\varphi \subset \text{mbl}''\,\psi$)
Step 3 ($0 \ne A \in \text{mbl}''\,\varphi \to .\psi A = .\psi A$)
Proof:
We can and do choose such an
$$(S \in \text{sqnc}(H\,\text{dmn}'\,\varphi))$$
that
$$(A \in \bigvee n \in \omega\,.\,Sn)\ .$$
Let
$$(P = \curlywedge n \in \omega(A\,.\,Sn \sim \bigvee \nu \in n\,.\,S\nu))$$
and note that
$$(A = \bigvee n \in \omega\,.\,Pn \wedge \bigwedge n \in \omega(.\,Pn \in \text{mbl}'\,\varphi \subset \text{mbl}\,\psi))\ .$$
Since 2.64.1 assures us
$$\bigwedge n \in \omega(.\,\psi\,.\,Pn = .\,\varphi\,.\,Pn)$$
we conclude
$$(.\,\psi A = \sum n \in \omega\,.\,\psi\,.\,Pn = \sum n \in \omega\,.\,\varphi\,.\,Pn = .\,\varphi A)$$
Step 4 $\bigwedge \beta \in \text{mbl}''\,\varphi(.\,\psi\beta = .\,\varphi\beta)$
Proof: Use Step 3.

2.66 ($\varphi \in \text{Msr}\,S \wedge H \in \nabla\text{field}\,\text{sb}\,\text{mbl}\,\varphi \wedge S = \nabla\,H \wedge \psi = \text{mss}\,\varphi S H \to$

.0 $\psi \in \text{Msr}\,S \wedge$

.1 $\bigwedge A \subset S(.\,\varphi A \le .\,\psi A) \wedge$

.2 $\bigwedge A \in H(.\,\varphi A = .\,\psi A) \wedge$

.3 $\psi \in \text{Hull}\,H \wedge$

.4 $H \subset \text{mbl}\,\psi \wedge$

.5 $\text{mbl}'\,\psi \subset \text{mbl}'\,\varphi \wedge$

.6 $\text{mbl}''\,\psi \subset \text{mbl}''\,\varphi \wedge$

.7 $\psi \in \text{Msh}$)

2.67 $(\varphi \in \mathrm{Msr}\, S \wedge \psi = \mathrm{om}\, \varphi \rightarrow$
.0 $\psi \in \mathrm{Msr}\, S \wedge$
.1 $\bigwedge A \subset S(\mathbf{.}\, \varphi A \leq \mathbf{.}\, \psi A) \wedge$
.2 $\bigwedge A \in \mathrm{mbl}\, \varphi(\mathbf{.}\, \varphi A = \mathbf{.}\, \psi A) \wedge$
.3 $\psi \in \mathrm{Hull}\, \mathrm{mbl}\, \varphi \wedge$
.4 $\mathrm{mbl}\, \varphi \subset \mathrm{mbl}\, \psi \wedge$
.5 $\mathrm{mbl}'\, \psi = \mathrm{mbl}'\, \varphi \wedge$
.6 $\mathrm{mbl}''\, \psi = \mathrm{mbl}''\, \varphi \wedge$
.7 $\psi \in \mathrm{Msh} \wedge$
.8 $(\mathrm{dmn}'\, \varphi \subset \mathrm{dmn}'\, \psi \rightarrow \mathrm{mbl}\, \varphi = \mathrm{mbl}\, \psi))$

2.68 $(\varphi \in \mathrm{Ms} \wedge \bigwedge \psi \in \mathrm{sms}\, \varphi(A \in \mathrm{mbl}\, \mathrm{om}\, \psi) \rightarrow A \in \mathrm{mbl}\, \varphi)$

2.69 Theorems
.0 $(0 \neq F \subset \mathrm{Msr}\, S \wedge \varphi = \curlywedge \beta \subset S \sup \psi \in F \mathbf{.}\, \psi\beta \rightarrow \varphi \in \mathrm{Msr}\, S)$
.1 $(\psi \in \mathrm{sqnc}\, \mathrm{Msr}\, S \wedge \varphi = \curlywedge \beta \subset S \sup n \in \omega \mathbf{..}\, \psi n \beta \rightarrow \varphi \in \mathrm{Msr}\, S)$

Veryify as many conclusions as you can in the following not too obvious theorem.
2.70 $(\varphi \in \mathrm{Msr}\, S \wedge \psi = (\mathrm{om}\, \varphi + \mathrm{im}\, \varphi)/2 \rightarrow$
.0 $\psi \in \mathrm{Msr}\, S \wedge$
.1 $(T \in \mathrm{mbl}\, \varphi \wedge A \subset S \rightarrow \mathbf{.}\, \psi T = \mathbf{.}\, \psi(TA) + \mathbf{.}\, \psi(T{\sim}A)) \wedge$
.2 $\mathrm{mbl}\, \mathrm{om}\, \varphi \subset \mathrm{mbl}\, \psi \wedge$
.3 $(\bigwedge M \in \mathrm{mbl}'\, \varphi \bigvee F \in \mathrm{cuv}\, M\, \mathrm{sb}\, S \bigwedge A, B, \in F(\mathbf{.}\, \mathrm{im}\, \varphi(A \cup B) = 0) \rightarrow \mathrm{mbl}\, \psi \subset \mathrm{mbl}\, \mathrm{om}\, \varphi))$
Theorem 2.70.3 was discovered and first proved by Trevor J. McMinn.

We now investigate add$''$ and some extensions of its members.
2.71 $(F'' \subset\subset F' \subset\subset F \rightarrow F'' \subset\subset F)$

2.72 $(\mathrm{grated\, is}\, H \wedge K = \mathrm{dsn}'\, H \wedge \alpha, \beta, \in K \rightarrow$
.0 $\alpha{\sim}\beta \in K \wedge$
.1 $\alpha \cap \beta \in K \wedge$
.2 $\alpha \cup \beta \in K)$
Hint: note first that $(A \in K \wedge B \in K \rightarrow A{\sim}B \in K)$.

Easy now is
2.73 $(\mathrm{grated\, is}\, H \rightarrow \mathrm{refined\, is}\, H)$

2.74 Theorems
.0 $(\mathrm{add}'' \subset \mathrm{add}')$
.1 $(g \in \mathrm{add}'\, \mathrm{gauge\, On}\, H \wedge \mathrm{grated\, is}\, H \rightarrow g \in \mathrm{subadd}'')$

2.75 $(g \in \mathrm{subadd}''\, \mathrm{gauge\, On}\, H \wedge \mathrm{grated\, is}\, H \wedge \varphi = \mathrm{mr}\, g \rightarrow H \subset \mathrm{mbl}\, \varphi)$
Proof
Because of 2.61 the desired conclusion is a consequence of the statement
$(A \in H \wedge T \in H \rightarrow \mathbf{.}\, gT \geq \mathbf{.}\, \varphi(TA) + \mathbf{.}\, \varphi(T{\sim}A))$.
Proof:
With the help of 2.72 we select G' and G'' that
$(G' \in \mathrm{fnt} \cap \mathrm{dsjn} \cap \mathrm{sb}\, H \wedge \nabla G' = TA \wedge$
$G'' \in \mathrm{fnt} \cap \mathrm{dsjn} \cap \mathrm{sb}\, H \wedge \nabla G'' = T{\sim}A)$.

Note that

$$(\nabla(G'{\sim}1) = TA \wedge \nabla(G''{\sim}1) = T{\sim}A \wedge (G'{\sim}1) \cap (G''{\sim}1) = 0)$$

and put

$$(G = (G'{\sim}1) \cup (G''{\sim}1)) \ .$$

Evidently

$$(G \in \mathrm{fnt} \cap \mathrm{dsjn} \cap \mathrm{sb}\, H \wedge \nabla\, G = T) \ .$$

Hence because of 2.53.2

$$\begin{aligned}(\boldsymbol{.}\, gT &\geq \textstyle\sum \beta \in G \,\boldsymbol{.}\, g\beta \\ &= \textstyle\sum \beta \in G' \backsim 1 \,\boldsymbol{.}\, g\beta + \sum \beta \in G'' \backsim 1 \,\boldsymbol{.}\, g\beta \\ &\geq \textstyle\sum \beta \in G' \backsim 1 \,\boldsymbol{.}\, \varphi\beta + \sum \beta \in G'' \backsim 1 \,\boldsymbol{.}\, g\varphi\beta \\ &\geq \boldsymbol{.}\, \varphi\, \nabla(G'{\sim}1) + \boldsymbol{.}\, \varphi(\nabla\, G''{\sim}1) \\ &= \boldsymbol{.}\, \varphi(TA) + \boldsymbol{.}\, \varphi(T{\sim}A)) \ .\end{aligned}$$

2.76 $(g \in \mathrm{add}''\, \mathrm{gauge}\, \mathrm{On}\, H \wedge \mathrm{grated}\, \mathrm{is}\, H \wedge \varphi = \mathrm{mr}\, g \rightarrow$

.0 $H \subset \mathrm{mbl}\, \varphi \wedge$

.1 $g \subset \varphi \in \mathrm{Msr}\, \nabla\, H)$

Proof:

We use 2.74 in checking

.2 $(g \in \mathrm{subadd}'')$

We infer .0 from .2 and 2.75 and infer .1 from 2.55, 2.63, and the

Statement $(A \in H \wedge F \in \mathrm{cuv}\, AH \rightarrow \boldsymbol{.}\, gA \leq \sum \beta \in F \,\boldsymbol{.}\, g\beta)$

Proof:

Let

$$(K = \mathrm{dmn}'\, H \wedge F' = \textstyle\bigvee \beta \in F\, \mathrm{sng}(\beta A))$$

With the help of 2.72.1 we easily check

$$(F' \in \mathrm{cbl}\, \mathrm{sb}\, K \wedge F' \subset\subset F \wedge \nabla\, F' = A)$$

Because of this and the principle of choice we so secure F'' that

$$(F'' \in \mathrm{cbl}\, \mathrm{sb}\, H \wedge F'' \subset\subset F' \wedge \nabla\, F'' = A)$$

Next because of 2.73 we choose

$$(G \in \mathrm{cbl}\, \mathrm{dsjn}\, \mathrm{sb}\, H \wedge G \subset\subset F'' \wedge \nabla\, G = A)$$

We notice

$$(G \subset\subset F'' \subset\subset F' \subset\subset F \wedge G \subset\subset F)$$

and because of 2.53.1, .2, and 2.53.2 we conclude

$$\begin{aligned}(\boldsymbol{.}\, gA &= \textstyle\sum \beta \in G \,\boldsymbol{.}\, g\beta \\ &= \textstyle\sum B \in \bigvee \beta \in F(G \cap \mathrm{sb}\, \beta) \,\boldsymbol{.}\, gB \\ &\leq \textstyle\sum \beta \in F \sum B \in G \cap \mathrm{sb}\, \beta \,\boldsymbol{.}\, gB \\ &\leq \textstyle\sum \beta \in F \,\boldsymbol{.}\, g\beta)\end{aligned}$$

2.77 $(g \in \mathrm{add}''\, \mathrm{gauge}\, \mathrm{On}\, H \wedge \mathrm{grated}\, \mathrm{is}\, H \wedge \varphi = \mathrm{mr}\, g \wedge H' \subset \mathrm{mbl}\, \varphi \wedge$

$g \subset g' \in \mathrm{add}''\, \mathrm{gauge}\, \mathrm{On}\, H' \wedge \mathrm{grated}\, \mathrm{is}\, H'$

$\rightarrow \bigwedge \beta \in H' \cap \mathrm{mbl}''\, \varphi(\boldsymbol{.}\, g'\beta = \boldsymbol{.}\, \varphi\beta))$

Proof:

Let $(S = \nabla\, H)$ and note that

$$(H \subset H' \wedge \nabla\, H = \nabla\, H' = S)$$

Next let $(\varphi' = \mathrm{mr}\, g')$. According to 2.76

$$(H' \subset \mathrm{mbl}\, \varphi \cap \mathrm{mbl}\, \varphi' \wedge g \subset g' \subset \varphi' \wedge g \subset \varphi \subset \varphi' \wedge \varphi' \in \mathrm{Msr}\, S \wedge \textstyle\bigwedge \beta \in H(\boldsymbol{.}\, \varphi'\beta = \boldsymbol{.}\, \varphi\beta))$$

According to this and 2.65

$$(\beta \in H' \cap \mathrm{mbl}''\, \varphi \rightarrow \boldsymbol{.}\, g\beta = \boldsymbol{.}\, \varphi'\beta = \boldsymbol{.}\, \varphi\beta)$$

We now look into metric constructions.

2.78 $(\rho\,\mathrm{metrizes}\,S \wedge \bigwedge n \in \omega(\mathbf{.}\,Fn = \mathrm{dmtr}\,\rho n \wedge \mathbf{.}\,\psi n = \mathrm{mss}\,gS\,\mathbf{.}\,Fn) \wedge \varphi = \mathrm{msm}\,g\rho H \rightarrow$

.0 $\varphi \in \mathrm{Md}\,\rho\,\wedge$

.1 $\bigwedge n \in \omega(\mathbf{.}\,\psi n \in \mathrm{Msr}\,S)\,\wedge$

.2 $\bigwedge n \in \omega \bigwedge A \subset S(\mathbf{..}\,\psi A \leq \mathbf{..}\,\psi(n+1)A)\,\wedge$

.3 $\varphi = \curlywedge A \subset S\,\sup n \in \omega\,\mathbf{..}\,\psi n A\,\wedge$

.4 $\bigwedge n \in \omega \bigwedge A \bigwedge B(\mathrm{dist}\,\rho AB > \underline{2}\,n \rightarrow \mathbf{..}\,\psi n(A \cup B) = \mathbf{..}\,\psi nA + \mathbf{..}\,\psi nB))$

Proof:

.1 follows from 2.55

.2 is a consequence of the fact that

$\bigwedge n \in \omega(\mathbf{.}\,Fn \supset \mathbf{.}\,F(n+1))$

.3 is a consequence of 2.5.1

.4 and .0 are consequences of 0, 1, and 2 below.

Statement 0 $(n \in \omega \wedge \mathrm{dist}\,\rho AB > \underline{2}\,n \rightarrow \mathbf{..}\,\psi n(A \cup B) = \mathbf{..}\,\psi nA + \mathbf{..}\,\psi nB)$

Proof:

We assume

$(\mathbf{..}\,\psi n(A \cup B) < \infty)$

since otherwise the desired conclusion is obvious. Let $(r > 0)$ and so 2.51.0 choose

$(F' \in \mathrm{cuv}(A \cup B)\,\mathbf{.}\,Fn)$

that

$(\sum \beta \in F'|\,\mathbf{.}\,g\beta| < \mathbf{..}\,\psi n(A \cup B) + r)$

Let

$(F'' = \mathsf{E}\,\beta \in F'(\beta A \neq 0) \wedge F''' = F' \sim F'')$

Since

$(F' \subset \mathbf{.}\,Fn = H\,\mathrm{dmtr}\,\rho n \wedge \mathrm{dist}\,\rho AB > \underline{2}\,n)$

we infer with the aid of 2.50.0 that

$(F''F''' = 0 \wedge F'' \cup F''' = F' \wedge A \subset \nabla F'' \wedge B \subset \nabla F''' \wedge F'' \in \mathrm{cuv}\,A\,\mathbf{.}\,Fn \wedge F''' \in \mathrm{cuv}\,B\,\mathbf{.}\,Fn)$

Consequently

$(\mathbf{..}\,\psi nA + \mathbf{..}\,\psi nB \leq \sum \beta \in F''|\,\mathbf{.}\,g\beta| + \sum \beta \in F'''|\,\mathbf{.}\,g\beta|$

$= \sum \beta \in F'|\,\mathbf{.}\,g\beta|$

$\leq \mathbf{..}\,\psi n(A \cup B) + r)$

Because of the arbitrary nature of r we now know

$(\mathbf{..}\,\psi nA + \mathbf{..}\,\psi nB \leq \mathbf{..}\,\psi n(A \cup B) \leq \mathbf{..}\,\psi nA + \mathbf{..}\,\psi nB)$

Statement 1 $(\mathrm{dist}\,\rho AB > 0 \rightarrow \mathbf{.}\,\varphi(A \cup B) = \mathbf{.}\,\varphi A + \mathbf{.}\,\varphi B)$

Proof:

From Statement 0 we gather that

$\mathrm{big}\,n(\mathbf{..}\,\psi n(A \cup B) = \mathbf{..}\,\psi nA + \mathbf{..}\,\psi nB)$

Hence because of .2 we now find

$(\mathbf{.}\,\varphi(A \cup B) = \mathrm{lin}\,n\,\mathbf{..}\,\psi n(A \cup B)$

$= \mathrm{lin}\,n(\mathbf{..}\,\psi nA + \mathbf{..}\,\psi nB)$

$= \mathrm{lin}\,n\,\mathbf{..}\,\psi nA + \mathrm{lin}\,n\,\mathbf{..}\,\psi nB$

$= \mathbf{.}\,\varphi A + \mathbf{.}\,\varphi B)$

Statement 2 $(\varphi \in \mathrm{Md}\,\rho)$

Proof:

Because of .1 and .3 and 2.67.1 we are sure

$(\varphi \in \mathrm{Msr}\,S)$

Statement 1 completes the proof.

2.79 $(\rho\,\text{metrizes}\,S \wedge H \subset H' \wedge \nabla H' = S \wedge \varphi = \text{msm}\,g\rho H \rightarrow \varphi \in \text{Md}\,\rho\,\text{Hull}\,\sqcap''\,\nabla''\,H')$

Proof:

We deduce our conclusion from 2.78 and the

Statement $(A \subset S \rightarrow \bigvee A' \in \sqcap''\,\nabla''\,H' \cap \text{sp}\,A(\centerdot\,\varphi A' = \centerdot\,\varphi A))$

Proof:

Suppose

$\bigwedge n \in \omega(\centerdot\,Fn = \text{dmtr}\,\rho n \wedge \centerdot\,\psi n = \text{mss}\,gS\,\centerdot\,Fn)$

Recall 2.60.2 and choose

$(S' \in \text{sqnc}\,\sqcap''\,\nabla''\,H')$

so that

$\bigwedge n \in \omega(A \subset \centerdot\,S'n \wedge \centerdot\centerdot\,\psi n\,\centerdot\,S'n = \centerdot\centerdot\,nA)$

Let

$(A' = \bigwedge n \in \omega\,\centerdot\,S'n)$

Clearly

$(A' \in \sqcap''\,\sqcap''\,\nabla''\,H' = \sqcap''\,\nabla''\,H')$

Furthermore

$$\begin{aligned}(A \subset A' \wedge \centerdot\,\varphi A &\leq \centerdot\,\varphi A' \\ &= \sup n \in \omega\,\centerdot\centerdot\,\psi n\,\centerdot\,A' \\ &\leq \sup n \in \omega\,\centerdot\centerdot\,\psi n\,\centerdot\,S'n \\ &= \sup n \in \omega\,\centerdot\centerdot\,\psi nA \\ &= \centerdot\,\varphi A)\end{aligned}$$

The proof is complete.

2.80 $(\varphi \in \text{Md}\,\rho \wedge S = \text{space}\,\rho \wedge \psi = \text{mss}\,\varphi S\,\text{open}\,\rho \rightarrow$

.0 $\psi \in \text{Md}\,\rho \wedge$

.1 $\bigwedge A \subset S(\centerdot\,\varphi A \leq \centerdot\,\psi A) \wedge$

.2 $\bigwedge A \in \text{open}\,\rho(\centerdot\,\varphi A = \centerdot\,\psi A) \wedge$

.3 $\bigwedge A \subset S \bigwedge r > 0 \bigvee B \in \text{open}\,\rho \cap \text{sp}\,A(\centerdot\,\psi B \leq \centerdot\,\psi A + r))$

Proof:
Because of 2.55

$(\psi \in \mathrm{Msr}\, S)$

In view of 2.5.9 and the fact that

$(S \in \mathrm{open}\, \rho)$

we see .3, .1, and .2 without difficulty. To establish .0, suppose

$(\mathrm{dist}\, \rho AB = K > 0)$

Let

$(A' = \bigvee x \in A\, \mathrm{sr}\, \rho x(K/3) \wedge A'' = \mathrm{intr}\, \rho A' \wedge B' = \bigvee x \in B\, \mathrm{sr}\, \rho x(K/3) \wedge B'' = \mathrm{intr}\, \rho B')$

Check that

$(A \subset A'' \wedge B \subset B'' \wedge A''B'' \subset A'B' = 0)$

Now

$$
\begin{aligned}
&(\alpha \in \mathrm{open}\, \rho \cap \mathrm{sp}(A \cup B) \rightarrow \\
&\quad A \subset \alpha A'' \in \mathrm{open}\, \rho\, \wedge \\
&\quad B \subset \alpha B'' \in \mathrm{open}\, \rho\, \wedge \\
&\quad .\varphi\alpha \geq .\varphi(\alpha A'' \cup \alpha B'') \\
&\qquad = .\varphi(\alpha A'') + .\varphi(\alpha B'') \\
&\qquad \geq .\psi A + .\psi B)
\end{aligned}
$$

Thus

$\bigwedge \alpha \in \mathrm{open}\, \rho \cap \mathrm{sp}(A \cup B)(.\varphi\alpha \geq .\psi A + .\psi B)$

Hence because of 2.59

$(.\psi(A \cup B) = \inf \alpha \in \mathrm{open}\, \rho \cap \mathrm{sp}(A \cup B)\, .\varphi\alpha \geq .\psi A + .\psi B)$

Consequently

$(.\psi(A \cup B) = .\psi A + .\psi B)$

and we are sure

$(\psi \in \mathrm{Md}\, \rho)$

Approximations

2.81 Definitions

.0 $(\mathrm{bore}\, \rho \equiv \mathsf{E}\, \beta(\rho \in \mathrm{metric} \wedge \bigwedge \varphi \in \mathrm{Md}\, \rho(\beta \in \mathrm{mbl}\, \varphi)))$

.1 $(\mathrm{Mh}\, \rho \equiv (\mathrm{Md}\, \rho \cap \mathrm{Hull\, bore}\, \rho))$

.2 $(\mathrm{Mbh}\, \rho \equiv \mathsf{E}\, \varphi \in \mathrm{Mh}\, \rho(\mathrm{bounded}\, \rho \subset \mathrm{dmn}'\, \varphi))$

2.82 Theorems

.0 $(F \in \mathrm{U} \wedge T \in \mathrm{To\, U} \wedge F' = \bigvee \beta \in F\, \mathrm{sng}\, {}_*T\beta \wedge B = \mathrm{Borel}\, F \wedge B' = \bigvee \beta \in B_* T\beta \rightarrow \mathrm{Borel}\, F' = B')$

.1 $(F \in \mathrm{U} \wedge F' = \bigvee \beta \in F\, \mathrm{sng}(\beta A) \wedge B = \mathrm{Borel}\, F \wedge B' = \bigvee \beta \in B\, \mathrm{sng}(\beta A) \rightarrow \mathrm{Borel}\, F' = B')$

Hint: let $(T = \curlywedge x \in A \cap \nabla Fx)$

2.83 Theorems

.0 $(\varphi \in \mathrm{Md}\, \rho \rightarrow \mathrm{bore}\, \rho \subset \mathrm{mbl}\, \varphi)$

.1 $(\varphi \in \mathrm{Mh}\, \rho \rightarrow \varphi \in \mathrm{Msh})$

.2 $(\rho \in \mathrm{metric} \rightarrow \mathrm{Borel\, closed}\, \rho \subset \mathrm{bore}\, \rho \in \nabla \mathrm{field})$

.3 $(\varphi \in \mathrm{Mh}\, \rho \wedge A \in \mathrm{bore}\, \rho \rightarrow \mathrm{sct}\, \varphi A \in \mathrm{Mh}\, \rho)$

2.84 Lemmas

.0 $(A \subset S \wedge A' = S{\sim}A \wedge A' \subset B \subset S \wedge B' = S{\sim}B \rightarrow B' \subset A \wedge A{\sim}B' = B{\sim}A')$

.1 $(A \subset S \wedge A' = S{\sim}A \wedge C \subset A' \wedge C' = S{\sim}C \rightarrow A \subset C' \wedge C'{\sim}A = A'{\sim}C)$

2.85 Theorems

.0 $(H \in \nabla\text{field} \wedge \varphi \in \text{Hull}\, H \wedge A \in \text{mbl}'\, \varphi \rightarrow \bigvee K \in H \cap \text{sp}\, A(A{\sim}K \in \text{zr}\, \varphi))$

Hint: let

$(S \in H \cap \text{sp}\, A \wedge .\varphi S = .\varphi A \wedge A' = S{\sim}A \wedge B' \in \text{hull}\, \varphi A' \cap H \wedge K = S{\sim}B')$

and use 2.84.0

.1 $(H \in \nabla\text{field} \wedge \varphi \in \text{Hull}\, H \wedge A \in \text{mbl}'\, \varphi \rightarrow \bigvee K \in H \cap \text{sb}\, A(A{\sim}K \in \text{zr}\, \varphi))$

Theorems

2.86 $(\varphi \in \text{Md}\, \rho \wedge A \in \text{bore}\, \rho \wedge A \subset \alpha \in \text{open}\, \rho\, \text{dmn}'\, \varphi \wedge r > 0 \rightarrow \bigvee B \in \text{open}\, \rho \cap \text{sp}\, A(.\varphi(B{\sim}A) \leq r))$

Proof: lineb Let

$(S = \text{space}\, \rho \wedge \psi = \text{mss}\, \varphi S\, \text{open}\, \rho)$.

Note that

$(.\psi A < \infty)$

and use 2.80.3 to secure such a

$(B \in \text{open}\, \rho \cap \text{sp}\, A)$

that

$(.\psi B \leq .\psi A + r)$.

Because of 2.80.0 and 2.81.0 we now know

$(A \in \text{mbl}'\, \psi)$

and hence because of 2.80.1

$.\varphi(B{\sim}A) \leq .\psi(B{\sim}A) = .\psi B - .\psi A \leq r)$.

2.87 $(\varphi \in \text{Mh}\, \rho \wedge A \subset \alpha \in \text{open}\, \rho\, \text{dmn}'\, \varphi \wedge r > 0 \rightarrow \bigvee B \in \text{open}\, \rho \cap \text{sp}\, A(.\varphi B \leq .\varphi A + r))$

2.88 $(\varphi \in \text{Md}\, \rho \wedge A \in \text{bore}\, \rho \cap \text{dmn}'\, \varphi \wedge r > 0 \rightarrow \bigvee C \in \text{closed}\, \rho \cap \text{sb}\, A(.\varphi(A{\sim}C) \leq r))$

Proof:

Let

$(S = \text{space}\, \rho \wedge A' = S{\sim}A \wedge \psi = \text{sct}\, \varphi A)$

We know from 2.46

$(\psi \in \text{Md}\, \rho)$

Moreover, since

$(.\psi S = .\varphi A < \infty)$

we can use 2.86 to secure such a

$(B \in \text{open}\, \rho \cap \text{sb}\, A')$

that

$(.\psi(B{\sim}A') \leq r)$

Letting

$(C = S{\sim}B)$

we see with the help of 2.84.0 that

$(C \in \text{closed}\, \rho \cap \text{sb}\, A \wedge .\varphi(A{\sim}C) = .\psi(A{\sim}C) = .\psi(B{\sim}A') \leq r)$

2.89 $(\varphi \in \text{Mh}\, \rho \wedge A \in \text{mbl}''\, \varphi \rightarrow \bigvee K \in \text{bore}\, \rho \cap \text{sb}\, A(A{\sim}K \in \text{zr}\, \varphi))$

Proof:

Let

$(H' = \text{bore}\, \rho)$

and use 2.83.2 and 2.85.1.

2.90 $(\varphi \in \text{Mh}\, \rho \wedge A \in \text{mbl}'\, \varphi \wedge r > 0 \rightarrow \bigvee C \in \text{closed}\, \rho \cap \text{sb}\, A(.\varphi(A{\sim}C) \leq r))$

2.91 $(\varphi \in \mathrm{Mbh}\,\rho \wedge A \in \mathrm{mbl}\,\varphi \wedge r > 0$
$\quad \to \bigvee C \in \mathrm{closed}\,\rho \cap \mathrm{sb}\,A \bigvee B \in \mathrm{open}\,\rho \cap \mathrm{sp}\,A(\mathbf{.}\,\varphi(A{\sim}C) \leq r \wedge \mathbf{.}\,\varphi(B{\sim}A) \leq r))$

2.92 $(\varphi \in \mathrm{Mbh}\,\rho \wedge A \in \mathrm{mbl}\,\varphi$
$\quad \to \bigvee K \in \nabla''\,\mathrm{closed}\,\rho \cap \mathrm{sb}\,A \bigvee L \in \sqcap''\,\mathrm{open}\,\rho \cap \mathrm{sp}\,A(A{\sim}K \cup L{\sim}A \in \mathrm{zr}\,\varphi))$

The Lebesgue Decomposition

2.93 Definitions

.0 $(\mathrm{conservative}\,H \equiv \mathsf{E}\,\psi \in \mathrm{Ms}(H \subset \mathrm{zr}\,\psi))$

.1 $(\mathrm{singular}\,H \equiv \mathsf{E}\,\theta \in \mathrm{Ms}(H \subset \mathrm{dmn}\,\theta \wedge \bigvee z \in H(\theta = \mathrm{sct}\,\theta z)))$

.2 $(\mathrm{cnsr}\,\varphi H \equiv \curlywedge A \in \mathrm{dmn}\,\varphi \inf \alpha \in H\,\mathbf{.}\,\varphi(A{\sim}\alpha))$

.3 $(\mathrm{sngr}\,\varphi H \equiv \curlywedge A \in \mathrm{dmn}\,\varphi \sup \alpha \in H\,\mathbf{.}\,\varphi(A\alpha))$

2.94 Theorems

.0 $(\varphi \in \mathrm{Msr}\,S \wedge 0 \neq H \wedge \psi = \mathrm{cnsr}\,\varphi H \wedge \theta = \mathrm{sngr}\,\varphi H \wedge A \subset S$
$\quad \to \mathrm{dmn}\,\psi = \mathrm{dmn}\,\theta = \mathrm{dmn}\,\varphi \wedge 0 \leq \mathbf{.}\,\psi A \leq \mathbf{.}\,\varphi A \wedge 0 \leq \mathbf{.}\,\theta A \leq \mathbf{.}\,\varphi A \wedge \mathbf{.}\,\varphi A \leq \mathbf{.}\,\psi A + \mathbf{.}\,\theta A)$

.1 $(\varphi \in \mathrm{Msr}\,S \cap \mathrm{conservative}\,H \wedge 0 \neq H \wedge \psi = \mathrm{cnsr}\,\varphi H \wedge \theta = \mathrm{cnsr}\,\varphi H \to \mathbf{.}\,\theta S = 0 \wedge \varphi = \psi)$

.2 $((\varphi \in \mathrm{Msr}\,S \cap \mathrm{singular}\,H \wedge 0 \neq H \wedge \psi = \mathrm{cnsr}\,\varphi H \wedge \theta = \mathrm{cnsr}\,\varphi H \to \mathbf{.}\,\psi S = 0 \wedge \varphi = \theta)$

.3 $(\varphi \in \mathrm{Msr}\,S \wedge \psi' \in \mathrm{conservative}\,H \cap \mathrm{Msr}\,S \wedge \theta' \in \mathrm{singular}\,H \cap \mathrm{Msr}\,S \wedge \varphi = \psi' + \theta'$
$\quad \to \psi' = \mathrm{cnsr}\,\varphi H \wedge \theta' = \mathrm{sngr}\,\varphi H)$

.4 $(\varphi \in \mathrm{Msr}\,S \wedge \nabla''\,H = H \subset \mathrm{mbl}\,\varphi \wedge \psi = \mathrm{cnsr}\,\varphi H \wedge \theta = \mathrm{sngr}\,\varphi H$
$\quad \to \psi \in \mathrm{conservative} \cap \mathrm{Msr}\,S \wedge \theta \in \mathrm{Msr}\,S \wedge \varphi = \psi + \theta \wedge \mathrm{mbl}\,\varphi = \mathrm{mbl}\,\psi\,\mathrm{mbl}\,\theta)$

.5 $(\varphi \in \mathrm{Msr}\,S \wedge \nabla''\,H = H \subset \mathrm{mbl}\,\varphi \wedge z, z', \in H \wedge z{\sim}z' \in \mathrm{zr}\,\varphi \wedge$
$\quad \psi = \mathrm{cnsr}\,\varphi H = \mathrm{sct}\,\varphi{\sim}z \wedge \theta = \mathrm{sngr}\,\varphi H = \mathrm{sct}\,\varphi z$
$\quad \to \psi = \mathrm{sct}\,\varphi{\sim}z' \wedge \theta = \mathrm{sct}\,\varphi z')$

Theorem

2.95 $(\varphi \in \mathrm{Msr}\,S \wedge \nabla''\,H = H \subset \mathrm{mbl}\,\varphi \wedge S \in \mathrm{mbl}''\,\varphi \wedge \psi = \mathrm{cnsr}\,\varphi H \wedge \theta = \mathrm{sngr}\,\varphi H \to$

.0 $\psi \in \mathrm{conservative}\,H \cap \mathrm{Msr}\,S \wedge \theta \in \mathrm{singular} \cap \mathrm{Msr}\,S \wedge$

.1 $\mathrm{mbl}\,\varphi = \mathrm{mbl}\,\psi = \mathrm{mbl}\,\theta \wedge$

.2 $\varphi = \psi + \theta \wedge$

.3 $\bigvee z \in H(\psi = \mathrm{sct}\,\varphi{\sim}z \wedge \theta = \mathrm{sct}\,\varphi z))$

In connection with 2.95 we should like to point out that

$(\varphi, \mu, \in \mathrm{Ms} \wedge H = \mathrm{mbl}\,\varphi\,\mathrm{zr}\,\mu \to \nabla''\,H = H \subset \mathrm{mbl}\,\varphi)$

Lebesgue Measure

2.96 Definitions

.0 $(\mathrm{Diam}\,A \equiv \mathrm{diam}\,\underline{\mathrm{rf}}\,A)$

.1 $(\mathcal{L} \equiv \mathrm{msm}\,\curlywedge\beta\,\mathrm{Diam}\,\beta\,\underline{\mathrm{rf}}\,\mathrm{open}\,\underline{\mathrm{rf}})$

.2 $(\mathrm{Lebesguemeasure} \equiv \mathcal{L})$

2.97 Lemmas

.0 $(\rho \in \text{metric} \wedge A \subset \text{space}\,\rho \wedge r > 0 \rightarrow \bigvee B \in \text{open}\,\rho \cap \text{sp}\,A(\text{diam}\,\rho B \leq \text{diam}\,\rho A + r))$

.1 $(A \subset \text{rf} \wedge r > 0 \rightarrow \bigvee B \in \text{open}\,\underline{\text{rf}} \cap \text{sp}\,A(\text{Diam}\,B \leq \text{Diam}\,A + r))$

.2 $(0 \neq A \subset \text{rf} \rightarrow \text{Diam}\,A = \text{Sup}\,A —\ \text{Inf}\,A)$

.3 $(a \in \text{rf} \wedge \beta \subset \text{rf} \wedge x, y, \in \beta \wedge a \leq x \leq y \rightarrow y — x + \text{Diam}(\beta\,\text{nt}\,ax) \leq \text{Diam}(\beta\,\text{nt}\,ay))$

As a matter of fact equality holds in .3.

Theorem

2.98 $(-\infty \leq a \leq b \leq \infty \wedge F \in \text{cuv}\,\text{nt}\,ab\,\text{open}\,\underline{\text{rf}} \rightarrow b — a \leq \sum \beta \in F\,\text{Diam}\,\beta)$

Proof:

Let

$(C = \mathsf{E}\, x \geq a(x — a \leq \sum \beta \in F\,\text{Diam}(\beta \cap \text{nt}\,ax)))$

and let

$(E = \text{Sup}\,C)$

A moment's thought convinces us the desired conclusion is a consequence of the

Statement. $\bigvee x \in C(b \leq x)$

Proof: (by contradiction)

Suppose

$\sim \bigvee x \in C(b \leq x)$

and note that

$(\bigwedge x \in C(x \leq b) \wedge E \leq b)$

However since obviously $(a \in C)$ we now know

$(a \leq E \leq b)$.

In as much as

$(F \in \text{cover}\,\text{nt}\,ab)$

we can and so so select $(\beta_0 \in F)$ that

$(E \in \beta_0)$.

Since

$(\beta_0 \in \text{open}\,\underline{\text{rf}})$

we next choose (x_0, y_0) so that

.0 $(x_0 \in \beta_0 \cap C \wedge E < y_0 \in \beta_0)$.

Clearly

$(x_0 \leq E < y_0 \wedge a \leq x_0 < y)$.

Letting

$(F' = F \sim \text{sng}\,\beta_0)$

and using .0 and 2.97.3 we infer

$$\begin{aligned}(&\textstyle\sum \beta \in F\,\text{Diam}(\beta\,\text{nt}\,ay_0)\\ &= \text{Diam}(\beta_0\,\text{nt}\,ay_0) + \textstyle\sum \beta \in F'\,\text{Diam}(\beta\,\text{nt}\,ay_0)\\ &\geq y_0 — x_0 + \text{Diam}(\beta_0\,\text{nt}\,ay_0) + \textstyle\sum \beta \in F'\,\text{Diam}(\beta\,\text{nt}\,ax_0)\\ &= (y_0 — x_0) + \textstyle\sum \beta \in F\,\text{Diam}(\beta\,\text{nt}\,ax_0)\\ &\geq (y_0 — x_0) + (x_0 — a)\\ &= y_0 — a)\end{aligned}$$

Accordingly

$(y_0 \in C \wedge y_0 \leq E)$

in contradiction to .0.

2.99 Lemmas

.0 $(\mathcal{L} \in \mathrm{Mh}\,\underline{\mathrm{rf}})$

.1 $(-\infty < a \leq b < \infty \to b - a \leq .\,\mathcal{L}\,\mathrm{nt}\,ab)$

.2 $(-\infty < a \leq b < \infty \to .\,\mathcal{L}\,\mathrm{nt}\,ab \leq b - a)$

2.100 Theorems

.0 $(\mathcal{L} \in \mathrm{Mbh}\,\underline{\mathrm{rf}} \wedge \bigwedge a, b, \in \mathrm{rf}(.\,\mathcal{L}\,\mathrm{nt}\,ab = |a - b|))$

.1 $(\varphi \in \mathrm{Mh}\,\underline{\mathrm{rf}} \wedge \bigwedge a, b, \in \mathrm{rf}(.\,\varphi\,\mathrm{nt}\,ab = |b - a|) \leftrightarrow \varphi = \mathcal{L})$

.2 $(\mathcal{L} = \mathrm{mss} \curlywedge \beta\,\mathrm{Diam}\,\beta\,\mathrm{rf\ open}\,\underline{\mathrm{rf}})$

.3 $(\mathcal{L} = \mathrm{mr} \curlywedge \beta\,\mathrm{Diam}\,\beta)$

In checking .1 it helps to be aware of the structure of sets in rf .

Chapter 3: Integration

3.0 Definitions

.0 $(\mathrm{rn} \equiv \bigvee p \in \omega' \bigvee q \in \omega \backsim 1\,\mathrm{sng}(p/q))$

.1 $(\mathrm{boel} \equiv \mathsf{E}\,\beta \subset \mathrm{rl}(\beta\,\mathrm{rf} \in \mathrm{bore}\,\underline{\mathrm{rf}}))$

.2 $(\mathrm{boelian} \equiv \mathsf{E}\,f \in \mathrm{Upon}\,\mathrm{rl} \cap \mathrm{To}\,\mathrm{rl} \bigwedge \lambda \in \mathrm{rl}(\mathsf{E}\,x(\boldsymbol{.}\,fx \geq \lambda) \in \mathrm{boel}))$

3.1 Definitions

.0 $(\mathrm{ru}\,Gx\underline{\mathrm{u}}x \equiv (G \in \nabla\mathrm{field} \wedge \bigwedge x \in \nabla G\,\mathrm{ril}\,\underline{\mathrm{u}}x))$

.1 $(\mathrm{uu}\,Gx\underline{\mathrm{u}}x \equiv (\mathrm{ru}\,Gx\underline{\mathrm{u}}x \wedge \bigwedge \lambda \in \mathrm{rl}(\mathsf{E}\,x \in \nabla G(\underline{\mathrm{u}}x \geq \lambda) \in G)))$

.2 $(\mathrm{ff}\,G \equiv \mathsf{E}\,f \in \mathrm{Upon}\,\nabla G \cap \mathrm{To}\,\mathrm{rl}\,\mathrm{uu}\,Gx\,\boldsymbol{.}\,fx)$

The members of ff mbl φ are frequently known as φ-measurable functions.

.3 $(\mathrm{nihl}\,G \equiv \mathsf{E}\,\beta(\mathrm{sb}\,\beta \subset G))$

.4 $(\mathrm{most}\,Gx\underline{\mathrm{u}}x \equiv (G \in \nabla\mathrm{field} \wedge \nabla G{\sim}\mathsf{E}\,x\underline{\mathrm{u}}x \in \mathrm{nihl}\,G))$

.5 $(\mathrm{ff}{+}\,G \equiv (\mathrm{ff}\,G \cap \mathrm{gauge} \cap \mathrm{On}\,\nabla G))$

In connection with .3 it is of some interest that

$$(G \in \nabla\mathrm{field} \rightarrow \nabla''\,\mathrm{nihl}\,G = \mathrm{nihl}\,G)$$

and that

$$(A \subset B \in \mathrm{nihl}\,G \rightarrow A \in \mathrm{nihl}\,G)$$

3.2 Definitions

.0 $(\mathrm{Alm}\,\varphi x \in A\underline{\mathrm{u}}x \equiv (\varphi \in \mathrm{Ms} \wedge A\,\mathrm{rlm}\,\varphi{\sim}\mathsf{E}\,x\underline{\mathrm{u}}x \in \mathrm{zr}\,\varphi))$

.1 $(\mathrm{Alm}\,\varphi x\underline{\mathrm{u}}x \equiv \mathrm{Alm}\,\varphi x \in \mathrm{rlm}\,\varphi\underline{\mathrm{u}}x)$

3.3 Definitions

.0 $(\mathrm{rail}\,y \equiv (\mathrm{ps}\,y - \mathrm{ng}\,y))$

.1 $(\mathrm{massable}\,\varphi x\underline{\mathrm{u}}x \equiv (\mathrm{uu}\,\mathrm{mbl}\,\varphi x\,\mathrm{rail}\,\underline{\mathrm{u}}x \wedge \mathrm{Alm}\,\varphi x\,\mathrm{ril}\,\underline{\mathrm{u}}x))$

.2 $(\mathrm{massile}\,\varphi x\underline{\mathrm{u}}x \equiv (\mathrm{massable}\,\varphi x\underline{\mathrm{u}}x \wedge \mathsf{E}\,x \in \mathrm{rlm}\,\varphi(\underline{\mathrm{u}}x \neq 0) \in \mathrm{mbl}''\,\varphi))$

.3 $(\mathrm{massile}{+}\,\varphi x\underline{\mathrm{u}}x \equiv (\mathrm{massile}\,\varphi x\underline{\mathrm{u}}x \wedge \mathrm{Alm}\,\varphi x(\underline{\mathrm{u}}x \geq 0)))$

Function Structure

Lemma

3.4 $(\mathrm{ru}\,Gx\underline{\mathrm{u}}x \wedge \mathrm{ru}\,Gx\underline{\mathrm{v}}x \wedge \mathrm{most}\,Gx\underline{\mathrm{u}}x(\underline{\mathrm{u}}x = \underline{\mathrm{v}}x) \wedge \mathrm{uu}\,Gx\underline{\mathrm{u}}x \rightarrow \mathrm{uu}\,Gx\underline{\mathrm{v}}x)$

Proof:

Let

$$(A = \mathsf{E}\,x \in \nabla G(\underline{\mathrm{u}}x = \underline{\mathrm{v}}x) \wedge B = \mathrm{set}\,of x \in \nabla G(\underline{\mathrm{u}}x \neq \underline{\mathrm{u}}x))$$

Clearly

$$(\bigwedge \alpha(B\alpha \in G) \wedge B \in G \wedge A = \nabla G{\sim}B \in G)$$

and hence

$$\begin{aligned}
&(\bigwedge \lambda(\lambda \in \mathrm{rl} \rightarrow \\
&\quad \mathsf{E}\,x \in \nabla G(\underline{\mathrm{v}}x \geq \lambda) \\
&\qquad = \mathsf{E}\,x \in A(\underline{\mathrm{v}}x \geq \lambda) \cup \mathsf{E}\,x \in B(\underline{\mathrm{v}}x \geq \lambda) \\
&\qquad = \mathsf{E}\,x \in A(\underline{\mathrm{u}}x \geq \lambda) \cup \mathsf{E}\,x \in B(\underline{\mathrm{v}}x \geq \lambda) \\
&\qquad \in G) \\
&\rightarrow \mathrm{uu}\,Gx\underline{\mathrm{v}}x)
\end{aligned}$$

We now have at once
Theorem
3.5 $(\mathrm{ru}\, Gx\underline{\mathrm{u}}x \wedge \mathrm{ru}\, Gx\underline{\mathrm{v}}x \wedge \mathrm{most}\, Gx(\underline{\mathrm{u}}x = \underline{\mathrm{v}}x) \rightarrow \mathrm{uu}\, Gx\underline{\mathrm{u}}x \leftrightarrow \mathrm{uu}\, Gx\underline{\mathrm{v}}x)$

3.6 Lemma
$(\lambda \in \mathrm{rl} \wedge D = \mathsf{E}\, x \in S(\underline{\mathrm{u}}x \in \mathrm{rl}) \wedge A \subset \mathsf{E}\, y(y > \lambda) \wedge B \subset \mathsf{E}\, y(y < \lambda) \wedge \mathrm{Inf}\, A = \lambda = \mathrm{Sup}\, B \rightarrow$
.0 $\mathsf{E}\, x \in S(\underline{\mathrm{u}}x > \lambda) = \bigvee y \in A\, \mathsf{E}\, x \in S(\underline{\mathrm{u}}x \geq y) \wedge$
.1 $\mathsf{E}\, x \in S(\underline{\mathrm{u}}x \leq \lambda) = D \sim \mathsf{E}\, x \in S(\underline{\mathrm{u}}x > \lambda) \wedge$
.2 $\mathsf{E}\, x \in S(\underline{\mathrm{u}}x < \lambda) = \bigvee y \in B\, \mathsf{E}\, x \in S(\underline{\mathrm{u}}x \leq y) \wedge$
.3 $\mathsf{E}\, x \in S(\underline{\mathrm{u}}x \geq \lambda) = D \sim \mathsf{E}\, x \in S(\underline{\mathrm{u}}x < \lambda))$

Lemma
3.7 $(\lambda \in \mathrm{rl} \wedge A = \mathsf{E}\, y \in \mathrm{rn}(y > \lambda) \wedge B = \mathsf{E}\, y \in \mathrm{rn}(y < \lambda)$
$\rightarrow A \in \mathrm{cbl} \wedge B \in \mathrm{cbl} \wedge A \subset \mathsf{E}\, y(y > \lambda) \wedge B \subset \mathsf{E}\, y(y < \lambda) \wedge \mathrm{Inf}\, A = \lambda = \mathrm{Sup}\, B)$

Lemma
3.8 $(\mathrm{uu}\, Gx\underline{\mathrm{u}}x \rightarrow \mathsf{E}\, x \in \nabla G(\underline{\mathrm{u}}x \in \mathrm{rl}) \in G)$
Proof:
$(\mathsf{E}\, x \in \nabla G(\underline{\mathrm{u}}x \in \mathrm{rl}) = \mathsf{E}\, x \in \nabla G(\underline{\mathrm{u}}x \geq -\infty) \in G)$

Theorem
3.9 $(S = \nabla G \wedge D = \mathsf{E}\, x \in S(\underline{\mathrm{u}}x \in \mathrm{rl}) \wedge D \in G \wedge \mathrm{ru}\, Gx\underline{\mathrm{u}}x \rightarrow$
.0 $\mathrm{uu}\, Gx\underline{\mathrm{u}}x \leftrightarrow$
.1 $\bigwedge \lambda \in \mathrm{rl}(\mathsf{E}\, x \in S(\underline{\mathrm{u}}x < \lambda) \in G) \leftrightarrow$
.2 $\bigwedge \lambda \in \mathrm{rl}(\mathsf{E}\, x \in S(\underline{\mathrm{u}}x \geq \lambda) \in G) \leftrightarrow$
.3 $\bigwedge \lambda \in \mathrm{rl}(\mathsf{E}\, x \in S(\underline{\mathrm{u}}x > \lambda) \in G) \leftrightarrow$
.4 $\bigwedge \lambda \in \mathrm{rl}(\mathsf{E}\, x \in S(\underline{\mathrm{u}}x \leq \lambda) \in G) \leftrightarrow$
.5 $\bigwedge \lambda \in \mathrm{rf}(\mathsf{E}\, x \in S(\underline{\mathrm{u}}x < \lambda) \in G) \leftrightarrow$
.6 $\bigwedge \lambda \in \mathrm{rf}(\mathsf{E}\, x \in S(\underline{\mathrm{u}}x \geq \lambda) \in G) \leftrightarrow$
.7 $\bigwedge \lambda \in \mathrm{rf}(\mathsf{E}\, x \in S(\underline{\mathrm{u}}x > \lambda) \in G) \leftrightarrow$
.8 $\bigwedge \lambda \in \mathrm{rf}(\mathsf{E}\, x \in S(\underline{\mathrm{u}}x \leq \lambda) \in G))$
Proof:
Obviously .0 is equivalent to .2. That .2 implies .3 is a consequence of 3.7 and 3.6.0.
That .3 implies .4 is a consequence of 3.6.1. That .4 implies .5 is a consequence of 3.7 and 3.6.2.
That .5 implies .6 is a consequence of 3.6.3. That .6 implies .7 is a consequence of 3.7 and 3.6.0.
That .7 implies .8 is a consequence of 3.6.1. That .8 implies .1 is a consequence of 3.7 and 3.6.2.
That .1 implies .2 is a consequence of 3.6.3.

Theorem
3.10 $(S = \nabla G \wedge \mathrm{ru}\, Gx\underline{\mathrm{u}}x \wedge \mathsf{E}\, x(\underline{\mathrm{u}}x = \infty) \in G \wedge \mathsf{E}\, x(\underline{\mathrm{u}}x = -\infty) \in G \wedge$
$\bigwedge \lambda \in \mathrm{rf}(\mathsf{E}\, x \in S(\lambda \leq x \in \mathrm{rf}) \in G)$
$\rightarrow \mathrm{uu}\, Gx\underline{\mathrm{u}}x)$

Theorem
3.11 $(\mathrm{uu}\,Gx\underline{\mathrm{u}}x \rightarrow$
 .0 $\mathrm{uu}\,Gx - \underline{\mathrm{u}}x \wedge$
 .1 $\mathrm{uu}\,Gx|\underline{\mathrm{u}}x| \wedge$
 .2 $\mathrm{uu}\,Gx\,\mathrm{ps}\,\underline{\mathrm{u}}x \wedge$
 .3 $\mathrm{uu}\,Gx\,\mathrm{ng}\,\underline{\mathrm{u}}x)$

Lemma
3.12 $(\mathrm{uu}\,Gx\underline{\mathrm{u}}x \wedge 0 \neq c \in \mathrm{rf} \rightarrow \mathrm{uu}\,Gx(c \cdot \underline{\mathrm{u}}x))$

Theorem
3.13 $(G \in \triangledown\mathrm{field} \wedge \mathrm{ril}\,c \rightarrow \mathrm{uu}\,Gxc)$

Theorem
3.14 $(\mathrm{uu}\,Gx\underline{\mathrm{u}}x \wedge \mathrm{uu}\,Gx\underline{\mathrm{v}}x \rightarrow \mathrm{uu}\,Gx(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x))$

Proof:
Let
$$(S = \triangledown\, G \wedge D = \mathsf{E}\, x \in S(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x \in \mathrm{rl}))$$
That
$$(D \in G)$$
follows from
$$(D = \mathsf{E}\, x((\underline{\mathrm{u}}x > -\infty \wedge \underline{\mathrm{v}}x > -\infty) \vee (\underline{\mathrm{u}}x < \infty \wedge \underline{\mathrm{v}}x < \infty)))$$
Now check that
$$(\lambda \in \mathrm{rf} \rightarrow \mathsf{E}\, x \in S(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x > \lambda) = \bigvee r \in \mathrm{rn}\,\mathsf{E}\, x \in S(\underline{\mathrm{u}}x > r \wedge \underline{\mathrm{v}}x > \lambda - r) \in G)$$

Lemma
3.15 $(\mathrm{uu}\,Gx\underline{\mathrm{u}}x \rightarrow \mathrm{uu}\,Gx(\underline{\mathrm{u}}x \cdot \underline{\mathrm{u}}x))$

Proof:
Let
$$(S = \triangledown\, G \wedge D = \mathsf{E}\, x \in S(\underline{\mathrm{u}}x \in \mathrm{rl}))$$
$$((\lambda < 0 \rightarrow \mathsf{E}\, x \in S(\underline{\mathrm{u}}x \cdot \underline{\mathrm{u}}x \geq \lambda) = D \in G) \wedge$$
$$(\lambda \geq 0 \rightarrow \mathsf{E}\, x \in S(\underline{\mathrm{u}}x \cdot \underline{\mathrm{u}}x \geq \lambda)$$
$$= \mathsf{E}\, x \in S \cap D(|\underline{\mathrm{u}}x| \cdot |\underline{\mathrm{u}}x| \geq \lambda)$$
$$= \mathsf{E}\, x \in S \cap D(|\underline{\mathrm{u}}x| \geq \sqrt{}\lambda) \in G))$$

Theorem
3.16 $(\mathrm{uu}\,Gx\underline{\mathrm{u}}x \wedge \mathrm{uu}\,Gx\underline{\mathrm{v}}x \rightarrow \mathrm{uu}\,Gx(\underline{\mathrm{u}}x \cdot \underline{\mathrm{v}}x))$

Proof:
Let $(S = \triangledown\, G)$.
$$(\mathsf{E}\, x \in S(\underline{\mathrm{u}}x \cdot \underline{\mathrm{v}}x = \infty)$$
$$= \mathsf{E}\, x \in S((\underline{\mathrm{u}}x = \infty \wedge \underline{\mathrm{v}}x > 0) \vee (\underline{\mathrm{u}}x = -\infty \wedge \underline{\mathrm{v}}x < 0) \vee$$
$$(\underline{\mathrm{u}}x > 0 \wedge \underline{\mathrm{v}}x = \infty) \vee (\underline{\mathrm{u}}x < 0 \wedge \underline{\mathrm{v}}x = -\infty)) \in G)$$
Similarly we see
$$(\mathsf{E}\, x \in S(\underline{\mathrm{u}}x \cdot \underline{\mathrm{v}}x = -\infty) \in G)$$
Now assume
$$\bigwedge x(\underline{\mathrm{w}}x = {}^{\bullet}((\underline{\mathrm{u}}x + \underline{\mathrm{v}}x)/2)2 - {}^{\bullet}((\underline{\mathrm{u}}x - \underline{\mathrm{v}}x)/2)2)$$
and use 3.14, 3.12, and 3.15 in checking
$$\mathrm{uu}\,Gx\underline{\mathrm{w}}x\ .$$

Now clearly

$$(\lambda \in \text{rf} \rightarrow \exists\, x \in S(\lambda \leq \underline{\text{u}}x \cdot \underline{\text{v}}x \in \text{rf})$$
$$= \exists\, x \in S(\lambda \leq \underline{\text{w}}x \in \text{rf}) \in G)$$

Invocation of 3.10 completes the proof.

Theorem

3.17 $(\text{uu}\, Gx\underline{\text{u}}x \wedge \text{uu}\, Gx\underline{\text{v}}x \rightarrow \text{uu}\, Gx(\underline{\text{u}}x \bullet \underline{\text{v}}x))$

Proof:

Let $(S = \nabla G)$.

$(\lambda > 0 \rightarrow \exists\, x \in S(\underline{\text{u}}x \bullet \underline{\text{v}}x \geq \lambda) = \exists\, x \in S(\underline{\text{u}}x \cdot \underline{\text{v}}x \geq \lambda))$

$(\lambda \leq 0 \rightarrow \exists\, x \in S(\underline{\text{u}}x \bullet \underline{\text{v}}x \geq \lambda) = \exists\, x \in S(\underline{\text{u}}x \cdot \underline{\text{v}}x \geq \lambda \vee \underline{\text{u}}x = 0 \vee \underline{\text{v}}x = 0))$

3.18 Theorems

.0 $(G \in \nabla\text{field} \rightarrow \text{uu}\, Gx\, \text{Cr}\, xA \leftrightarrow A\, \nabla\, G \in G)$

.1 $(\text{uu}\, \text{mbl}\, \varphi x\, \text{Cr}\, xA \leftrightarrow A\, \text{rlm}\, \varphi \in \text{mbl}\, \varphi)$

3.19 Theorems

.0 $(\text{ril}\, c \wedge \text{uu}\, Gx\underline{\text{u}}x \rightarrow \text{uu}\, Gx(c \cdot \underline{\text{u}}x))$

.1 $(\text{ril}\, c \wedge \text{uu}\, Gx\underline{\text{u}}x \rightarrow \text{uu}\, Gx(c \bullet \underline{\text{u}}x))$

3.20 Theorems

.0 $(\varphi \in \text{Ms} \rightarrow \text{zr}\, \varphi \subset \text{nihl}\, \text{mbl}\, \varphi)$

.1 $(\text{Alm}\, \varphi x\underline{\text{u}}x \rightarrow \text{most}\, \text{mbl}\, \varphi x\underline{\text{u}}x)$

.2 $(\varphi \in \text{Ms} \wedge \bigwedge x \in \text{rlm}\, \varphi\underline{\text{u}}x \rightarrow \text{Alm}\, \varphi x\underline{\text{u}}x)$

.3 $(\text{Alm}\, \varphi x\underline{\text{u}}x \wedge \text{Alm}\, \varphi x\underline{\text{v}}x \leftrightarrow \text{Alm}\, \varphi x(\underline{\text{u}}x \wedge \underline{\text{v}}x))$

.4 $(K \in \text{cbl} \wedge \varphi \in \text{Ms} \rightarrow \bigwedge n \in K\, \text{Alm}\, \varphi x\underline{\text{u}}'xn \leftrightarrow \text{Alm}\, \varphi x \bigwedge n \in K\underline{\text{u}}'xn)$

.5 $(\bigwedge n \in \omega\, \text{Alm}\, \varphi x\underline{\text{u}}'xn \leftrightarrow \text{Alm}\, \varphi x \bigwedge n \in \omega\underline{\text{u}}'xn)$

.6 $(\text{big}\, n\, \text{Alm}\, \varphi x\underline{\text{u}}'xn \rightarrow \text{Alm}\, \varphi x\, \text{big}\, n\underline{\text{u}}'xn)$

Theorems

3.21 $(\text{ru}\, \text{mbl}\, \varphi x\underline{\text{u}}x \leftrightarrow \varphi \in \text{Ms} \wedge \bigwedge x \in \text{rlm}\, \varphi\, \text{ril}\, \underline{\text{u}}x)$

3.22 $(\text{ru}\, \text{mbl}\, \varphi x\underline{\text{u}}x \wedge \text{ru}\, \text{mbl}\, \varphi x\underline{\text{v}}x \wedge \text{Alm}\, \varphi x(\underline{\text{u}}x = \underline{\text{v}}x) \rightarrow \text{uu}\, \text{mbl}\, \varphi x\underline{\text{u}}x \leftrightarrow \text{uu}\, \text{mbl}\, \varphi x\underline{\text{v}}x)$

3.23 $(A \in \text{open}\, \underline{\text{rf}} \wedge B = \text{rf} \sim A \rightarrow A = \bigvee r \in \text{rn} \exists\, t(\text{Sup}(B\, \text{nt} - \infty r) < t < \text{Inf}(B\, \text{nt}\, r\infty)))$

3.24 $(f \in \text{ff}\, G \wedge A \in \text{open}\, \underline{\text{rf}} \rightarrow {*}fA \in G)$

3.25 $(\rho \in \text{metric} \wedge f \in \text{Upon}\, \text{rlm}\, \varphi \wedge \bigwedge \beta \in \text{open}\, \rho({*}f\beta \in \text{mbl}\, \varphi) \rightarrow \bigwedge \beta \in \text{bore}\, \rho({*}f\beta \in \text{mbl}\, \varphi))$

Proof:

Let

$$(S = \text{rlm}\, \varphi \wedge R = \text{space}\, \rho \wedge S' = {*}fR)$$

Clearly
.0 $(S' \in \mathrm{mbl}\,\varphi)$
Suppose
$$(\varphi' \in \mathrm{sms}\,\mathrm{sct}\,\varphi S' \wedge \psi = \curlywedge\beta \subset R \,\textbf{.}\, \varphi' {}^*f\beta)$$
After checking that
$$(\psi \in \mathrm{Msr}\, R)$$
we proceed to show
$$(\psi \in \mathrm{Md}\,\rho)\ .$$
$$\begin{aligned}
&(T \subset R \wedge \beta \in \mathrm{open}\,\rho \rightarrow {}^*f\beta \in \mathrm{mbl}\,\varphi \subset \mathrm{mbl}\,\varphi' \rightarrow \\
&\quad \textbf{.}\,\psi(T\beta) + \textbf{.}\,\psi(T{\sim}\beta) \\
&\qquad = \textbf{.}\,\varphi'{}^*f(T\beta) + \textbf{.}\,\varphi'{}^*f(T{\sim}\beta) \\
&\qquad = \textbf{.}\,\varphi'({}^*fT{}^*f\beta) + \textbf{.}\,\varphi'({}^*fT{\sim}{}^*f\beta) \\
&\qquad = \textbf{.}\,\varphi'{}^*fT \\
&\qquad = \textbf{.}\,\psi T)
\end{aligned}$$
Consequently
$$(\mathrm{open}\,\rho \subset \mathrm{mbl}\,\psi)$$
and according to 2.48 it follows that
$$(\psi \in \mathrm{Md}\,\rho)\ .$$
Next because of 2.81.0 and 2.81.1
$$\begin{aligned}
&(\beta \in \mathrm{bore}\,\rho \rightarrow \beta \in \mathrm{mbl}\,\psi \rightarrow \\
&\quad (\textbf{.}\,\varphi' S \\
&\qquad = \textbf{.}\,\varphi' S' \\
&\qquad = \textbf{.}\,\psi R \\
&\qquad = \textbf{.}\,\psi\beta + \textbf{.}\,\psi(R{\sim}\beta) \\
&\qquad = \textbf{.}\,\varphi'{}^*f\beta + \textbf{.}\,\varphi'(S'{\sim}{}^*f\beta) \\
&\qquad = \textbf{.}\,\varphi'{}^*f\beta = \textbf{.}\,\varphi'(S{\sim}{}^*f\beta))
\end{aligned}$$
Thus because of the arbitrary nature of φ' we can now infer with the help of .0 and 2.11.1 that
$$\begin{aligned}
&(\beta \in \mathrm{bore}\,\rho \\
&\quad \rightarrow {}^*f\beta \in \mathrm{mbl}\,\mathrm{sct}\,\varphi S' \\
&\quad \rightarrow {}^*f\beta = {}^*f\beta S' \in \mathrm{mbl}\,\varphi \\
&\quad \rightarrow {}^*f\beta \in \mathrm{mbl}\,\varphi)
\end{aligned}$$

From 3.24 and 3.25 we deduce
3.26 $(f \in \mathrm{ff}\,\mathrm{mbl}\,\varphi \rightarrow \bigwedge \beta \in \mathrm{bore}\,\underline{\mathrm{rf}}({}^*f\beta \in \mathrm{mbl}\,\varphi))$

Lemma
3.27 $(f \in \mathrm{ff}\,\mathrm{mbl}\,\varphi \wedge y \in \mathrm{rl} \rightarrow {}^*f(\beta\,\mathrm{sng}\,y) \in \mathrm{mbl}\,\varphi)$

Theorems
3.28 $(f \in \mathrm{ff}\,\mathrm{mbl}\,\varphi \rightarrow \bigwedge \beta \in \mathrm{boel}({}^*f\beta \in \mathrm{mbl}\,\varphi))$
Proof:
Recalling 3.0.1 and using 3.26 and 3.27 we infer
$$\begin{aligned}
&(\beta \in \mathrm{boel} \rightarrow \\
&\quad {}^*f\beta \\
&\qquad = {}^*f(\beta\,\mathrm{rf} \cup \beta\,\mathrm{sng}\,\infty \cup \beta\,\mathrm{sng} - \infty) \\
&\qquad = {}^*f(\beta\,\mathrm{rf}) \cup {}^*f(\beta\,\mathrm{sng}\,\infty) \cup {}^*f(\beta\,\mathrm{sng} - \infty) \\
&\qquad \in \mathrm{mbl}\,\varphi)
\end{aligned}$$

3.29 $(\mathrm{boel} \in \nabla \mathrm{field})$

3.30 $(f \in \mathrm{boelian} \to \bigwedge \beta \in \mathrm{boel}(*f\beta \in \mathrm{boel}))$

Proof:

Let

$(g = \curlywedge x \in \mathrm{rf} \boldsymbol{.} fx)$

and note that

$.0 \bigwedge \alpha(*g\alpha = \mathrm{rf} * f\alpha)$

and that

$(\lambda \in \mathrm{rl} \to \mathsf{E}\, x \in \mathrm{rf}(\boldsymbol{.} gx \geq \lambda) = \mathsf{E}\, x \in \mathrm{rf}(\boldsymbol{.} fx \geq \lambda) \in \mathrm{bore}\, \underline{\mathrm{rf}})$

Hence

$(\lambda \in \mathrm{rl} \wedge \varphi \in \mathrm{Md}\, \underline{\mathrm{rf}} \to \mathsf{E}\, x \in \mathrm{rf}(\boldsymbol{.} gx \geq \lambda) \in \mathrm{mbl}\, \varphi)$

$(\varphi \in \mathrm{Md}\, \underline{\mathrm{rf}} \to g \in \mathrm{ff}\, \mathrm{mbl}\, \varphi)$

Because of 3.28 we now infer with the help of .0 that

$(\varphi \in \mathrm{Md}\, \underline{\mathrm{rf}} \wedge \beta \in \mathrm{boel} \to *g\beta \in \mathrm{mbl}\, \varphi)$

and

$(\beta \in \mathrm{boel} \to *g\beta \in \mathrm{bore}\, \underline{\mathrm{rf}})$

and

$(\beta \in \mathrm{boel} \to *f\beta \in \mathrm{boel})$

3.31 $(f \in \mathrm{boelian} \wedge g \in \mathrm{ff}\, \mathrm{mbl}\, \varphi \to f : g \in \mathrm{ff}\, \mathrm{mbl}\, \varphi)$

Proof:

$(\lambda \in \mathrm{rl} \wedge \beta = \mathsf{E}\, x(\boldsymbol{.} fx \geq \lambda)$

$\to \beta \in \mathrm{boel}$

$\to *g\beta \in \mathrm{mbl}\, \varphi$

$\to \mathsf{E}\, t(\boldsymbol{.}(f : g)t \geq \lambda) = *g\beta \in \mathrm{mbl}\, \varphi)$

Accordingly

$(f : g \in \mathrm{ff}\, \mathrm{mbl}\, \varphi)$

3.32 $(f \in \mathrm{boelian} \wedge g \in \mathrm{boelian} \to f : g \in \mathrm{boelian})$

3.33 $(f \in \mathrm{To}\, \mathrm{rl} \wedge \mathrm{dmn}\, f \in \mathrm{boel} \wedge A = \mathrm{rf}\, \mathrm{dmn}\, f \wedge \rho = \curlywedge x, y, \in A|x - y| \wedge$
$g = \curlywedge x \in A \boldsymbol{.} fx \wedge g \in \mathrm{Continuous}\, \rho\, \underline{\mathrm{rf}}$
$\to f \in \mathrm{boelian})$

Hint: $(C \in \mathrm{closed}\, \rho \leftrightarrow C = A\, \mathrm{clsr}\, \underline{\mathrm{rf}}\, C)$

3.34 $(G \in \nabla \mathrm{field} \wedge K \in \mathrm{cbl} \wedge \bigwedge n \in K\, \mathrm{uu}\, Gx\underline{\mathrm{u}}'xn \to \mathrm{uu}\, Gx \inf n \in K\underline{\mathrm{u}}'xn)$

Proof:

Let

$(S = \nabla G)$.

We have

$(\lambda \in \mathrm{rl} \to$

$\mathsf{E}\, x \in S(\inf n \in K\underline{\mathrm{u}}'xn \geq \lambda)$

$= \mathsf{E}\, x \in S \bigwedge n \in K(\underline{\mathrm{u}}'xn \geq \lambda)$

$= S \bigwedge n \in K\, \mathsf{E}\, x(\underline{\mathrm{u}}'xn \geq \lambda)$

$\in G)$

3.35 $(G \in \nabla\text{field} \wedge K \in \text{cbl} \wedge \bigwedge n \in K\ \text{uu}\, Gx\underline{\text{u}}'xn \to \text{uu}\, Gx \sup n \in K\underline{\text{u}}'xn)$

Proof:

$(\sup n \in K\underline{\text{u}}'xn = -\inf n \in K - \underline{\text{u}}'xn)$

3.36 $(\bigwedge n \in \omega\ \text{uu}\, Gx\underline{\text{u}}'xn \to \text{uu}\, Gx\, \underline{\text{lin}}\, n\underline{\text{u}}'xn)$

Proof:

Note that

$(\mathsf{E}\, x(\underline{\text{lin}}\, n\underline{\text{u}}'xn \geq \lambda) = \bigvee N \in \omega\, \mathsf{E}\, x(\lambda \leq \sup m \in \omega \backsim N \inf n \in \omega \backsim m\underline{\text{u}}'xn))$

3.37 $(\bigwedge n \in \omega\ \text{uu}\, Gx\underline{\text{u}}'xn \to \text{uu}\, Gx\, \overline{\text{lin}}\, n\underline{\text{u}}'xn)$

3.38 $(\bigwedge n \in \omega\ \text{uu}\, Gx\underline{\text{u}}'xn \to \text{uu}\, Gx\, \text{lin}\, n\underline{\text{u}}'xn)$

Proof:

$(\lambda \in \text{rl} \to \mathsf{E}\, x(\text{lin}\, n\underline{\text{u}}'xn \geq \lambda) = \mathsf{E}\, x(\underline{\text{lin}}\, n\underline{\text{u}}'xn \geq \lambda) \sim \mathsf{E}\, x(\overline{\text{lin}}\, n\underline{\text{u}}'xn \text{ --- } \underline{\text{lin}}\, n\underline{\text{u}}'xn > 0))$

3.39 $(G \in \nabla\text{field} \wedge K \in \text{cbl} \wedge \bigwedge n \in K\ \text{uu}\, Gx\underline{\text{u}}'xn \to \text{uu}\, Gx \sum n \in K\underline{\text{u}}'xn)$

3.40 $(\varphi \in \text{Md}\, \rho \wedge A \in \text{mbl}\, \varphi \wedge \rho' = \curlywedge x, y, \in A \centerdot \rho(x, y) \wedge f \in \text{Continuous}\, \rho'\, \underline{\text{rf}} \to f \in \text{ff mbl}\, \varphi)$

Proof:

After checking that

$(C \in \text{closed}\, \rho' \leftrightarrow C = A\, \text{clsr}\, \rho C)$

we see that

$(\lambda \in \text{rl} \to \mathsf{E}\, x(\centerdot\, fx \geq \lambda) \in \text{closed}\, \rho' \subset \text{mbl}\, \varphi)$

3.41 Theorems

.0 $(\text{rail}\, y = \text{ps}\, y \text{ --- } \text{ng}\, y)$

.1 $(\text{Alm}\, \varphi x(\underline{\text{u}}x = \underline{\text{v}}x) \to \text{massable}\, \varphi x\underline{\text{u}}x \leftrightarrow \text{massable}\, \varphi x\underline{\text{v}}x)$

.2 $(\text{massable}\, \varphi x\underline{\text{u}}x \wedge \text{massable}\, \varphi x\underline{\text{v}}x \to$

$\text{massable}\, \varphi x(\underline{\text{u}}x + \underline{\text{v}}x) \wedge$

$\text{massable}\, \varphi x(\underline{\text{u}}x \cdot \underline{\text{v}}x) \wedge$

$\text{massable}\, \varphi x(\underline{\text{u}}x \bullet \underline{\text{v}}x))$

.3 $(\varphi \in \text{Ms} \wedge \text{ril}\, c \to \text{massable}\, \varphi xc)$

.4 $(\text{ril}\, c \wedge \text{massable}\, \varphi c\underline{\text{u}}x \to \text{massable}\, \varphi x(c \cdot \underline{\text{u}}x) \wedge \text{massable}\, \varphi x(c \bullet \underline{\text{u}}x))$

.5 $(\text{massable}\, \varphi x\underline{\text{u}}x \to \text{massable}\, \varphi x\, \text{ps}\, \underline{\text{u}}x \wedge \text{massable}\, \varphi x\, \text{ng}\, \underline{\text{u}}x \wedge \text{massable}\, \varphi x|\underline{\text{u}}x|)$

.6 $(\text{massable}\, \varphi x\, \text{Cr}\, xA \leftrightarrow A\, \text{rlm}\, \varphi \in \text{mbl}\, \varphi)$

.7 $(\varphi \in \text{Ms} \wedge K \in \text{cbl} \wedge \bigwedge n \in K\ \text{massable}\, \varphi x\underline{\text{u}}'xn \to$

$\text{massable}\, \varphi x \inf n \in K\underline{\text{u}}'xn \wedge$

$\text{massable}\, \varphi x \sup n \in K\underline{\text{u}}pxn \wedge$

$\text{massable}\, \varphi x \sum n \in K\underline{\text{u}}'xn)$

.8 $(\bigwedge n \in \omega\ \text{massable}\, \varphi x\underline{\text{u}}'xn \to$

$\text{massable}\, \varphi x\, \underline{\text{lin}}\, n\underline{\text{u}}'xn \wedge$

$\text{massable}\, \varphi x\, \overline{\text{lin}}\, n\underline{\text{u}}'xn \wedge$

$\text{massable}\, \varphi x\, \text{lin}\, n\underline{\text{u}}'xn)$

.9 $(\text{uu mbl}\, \varphi x\underline{\text{u}}x \to \text{massable}\, \varphi x\underline{\text{u}}x)$

3.42 Theorems

.1 $(\text{Alm}\,\varphi x(\underline{\text{u}}x = \underline{\text{v}}x) \rightarrow \text{massile}\,\varphi x\underline{\text{u}}x \leftrightarrow \text{massile}\,\varphi x\underline{\text{v}}x)$

.2 $(\text{massable}\,\varphi x\underline{\text{u}}x \wedge \text{massile}\,\varphi x\underline{\text{v}}x \rightarrow \text{massile}\,\varphi x(\underline{\text{u}}x \bullet \underline{\text{v}}x))$

.3 $(\text{ril}\,c \wedge \text{massile}\,\varphi x\underline{\text{u}}x \rightarrow \text{massile}\,\varphi x(c \bullet \underline{\text{u}}x))$

.4 $(c \in \text{rf} \wedge \text{massile}\,\varphi x\underline{\text{u}}x \rightarrow \text{massile}\,\varphi x(c \cdot \underline{\text{u}}x))$

.5 $(\text{massile}\,\varphi x\underline{\text{u}}x \rightarrow \text{massile}\,\varphi x\,\text{ps}\,\underline{\text{u}}x \wedge \text{massile}\,\varphi x\,\text{ng}\,\underline{\text{u}}x \wedge \text{massile}\,\varphi x|\underline{\text{u}}x|)$

.6 $(\text{massile}\,\varphi x\underline{\text{u}}x \wedge \text{massile}\,\varphi x\underline{\text{v}}x \rightarrow$

$\text{massile}\,\varphi x(\underline{\text{u}}x + \underline{\text{v}}x) \wedge$

$\text{massile}\,\varphi x(\underline{\text{u}}x \cdot \underline{\text{v}}x) \wedge$

$\text{massile}\,\varphi x(\underline{\text{u}}x \bullet \underline{\text{v}}x))$

.7 $(\text{massable}\,\varphi x\underline{\text{u}}x \wedge \text{rlm}\,\varphi \in \text{mbl}''\,\varphi \rightarrow \text{massile}\,\varphi x\underline{\text{u}}x)$

.8 $(\text{massile}\,\varphi x\,\text{Cr}\,xA \leftrightarrow A\,\text{rlm}\,\varphi \in \text{mbl}''\,\varphi)$

.9 $(\text{massile}\,\varphi x\underline{\text{u}}x \wedge A \in \text{mbl}\,\varphi \rightarrow \text{massile}\,\varphi x(\underline{\text{u}}x \bullet \text{Cr}\,xA) \wedge \text{massile}\,\varphi x(\underline{\text{u}}x \cdot \text{Cr}\,xA))$

.10 $(\text{massable}\,\varphi x\underline{\text{u}}x \wedge \text{massile}\,\varphi x\underline{\text{v}}x \wedge \text{Alm}\,\varphi x(|\underline{\text{u}}x| \leq |\underline{\text{v}}x|) \rightarrow \text{massile}\,\varphi x\underline{\text{u}}x)$

.11 $(0 \neq K \in \text{cbl} \wedge \bigwedge n \in K\,\text{massile}\,\varphi x\underline{\text{u}}'xn \rightarrow$

$\text{massile}\,\varphi x\,\text{inf}\,n \in K\underline{\text{u}}'xn \wedge$

$\text{massile}\,\varphi x\,\text{sup}\,n \in K\underline{\text{u}}'xn)$

Hint: $(\text{rail}\,\text{inf}\,n \in K\underline{\text{u}}'xn = \text{inf}\,n \in K\,\text{rail}\,\underline{\text{u}}'xn)$

.12 $(\varphi \in \text{Ms} \wedge K \in \text{cbl} \wedge \bigwedge n \in K\,\text{massile}\,\varphi x\underline{\text{u}}'xn \rightarrow \text{massile}\,\varphi x \sum n \in K\underline{\text{u}}'xn)$

.13 $(\bigwedge n \in \omega\,\text{massile}\,\varphi x\underline{\text{u}}'xn \rightarrow \text{massile}\,\varphi x\,\underline{\text{lin}}\,n\underline{\text{u}}'xn \wedge \text{massile}\,\varphi x\,\overline{\text{lin}}\,n\underline{\text{u}}'xn \wedge \text{massile}\,\varphi x\,\text{lin}\,n\underline{\text{u}}'xn)$

.14 $(\text{Alm}\,\varphi x(\underline{\text{u}}x = \underline{\text{v}}x) \rightarrow \text{massile}\,\varphi x\underline{\text{u}}x \leftrightarrow \text{massile}\,\varphi x\underline{\text{v}}x)$

.15 $(\text{massable}\,\varphi x\underline{\text{u}}x \wedge \text{Alm}\,\varphi x(\underline{\text{u}}x \geq 0) \wedge \text{massile+}\,\varphi x\underline{\text{v}}x \rightarrow \text{massile+}\,\varphi x(\underline{\text{u}}x + \underline{\text{v}}x))$

.16 $(\text{rilp}\,c \wedge \text{massile+}\,\varphi x\underline{\text{u}}x \rightarrow \text{massile+}\,\varphi x(c \bullet \underline{\text{u}}x))$

.17 $(0 < c < \infty \wedge \text{massile+}\,\varphi x\underline{\text{u}}x \rightarrow \text{massile+}\,\varphi x(c \cdot \underline{\text{u}}x))$

.18 $(\text{massile+}\,\varphi x\underline{\text{u}}x \wedge \text{massile+}\,\varphi x\underline{\text{v}}x \rightarrow \text{massile+}\,\varphi x(\underline{\text{u}}x + \underline{\text{v}}x) \wedge \text{massile+}\,\varphi x(\underline{\text{u}}x \bullet \underline{\text{v}}x))$

.19 $(0 \neq K \in \text{cbl} \wedge \bigwedge n \in K\,\text{massile+}\,\varphi x\underline{\text{u}}'xn \rightarrow$

$\text{massile+}\,\varphi x\,\text{inf}\,n \in K\underline{\text{u}}'xn \wedge$

$\text{massile+}\,\varphi x\,\text{sup}\,n \in K\underline{\text{u}}'xn)$

.20 $(\varphi \in \text{Ms} \wedge K \in \text{cbl} \wedge \bigwedge n \in K\,\text{massile+}\,\varphi x\underline{\text{u}}'xn \rightarrow \text{massile+}\,\varphi x \sum n \in K\underline{\text{u}}'xn)$

.21 $(\bigwedge n \in \omega\,\text{massile+}\,\varphi x\underline{\text{u}}'xn \rightarrow \text{massile+}\,\varphi x\,\underline{\text{lin}}\,n\underline{\text{u}}'xn \wedge \text{massile+}\,\varphi x\,\overline{\text{lin}}\,n\underline{\text{u}}'xn)$

Schemes of Subdivision

3.43 Definitions

.0 $(\text{sng}'\,x \equiv (\sim\!1\,\text{sng}\,x))$

.1 $((D \,\underline{\cap}\, D') \equiv \bigvee \alpha \in D \bigvee \alpha' \in D'\,\text{sng}'(\alpha \cap \alpha'))$

.2 $(\text{scheme}\,\varphi \equiv \text{E}\,D \in \text{dsjn} \cap \text{cuv}\,\text{rlm}\,\varphi(\sim\!1\,\text{mbl}\,\varphi)(\varphi \in \text{Ms}))$

It may be helpful to recall that

$(\sim\!1 = \text{E}\,x(x \neq 0))$

3.44 Theorems

.0 $(D \sqcap D' = D' \sqcap D \subset \sim 1)$

.1 $(D \sqcap (D' \sqcap D'') = (D \sqcap D') \sqcap D'')$

.2 $(D \sqcap (D' \cup D'') = (D \sqcap D') \cup (D \sqcap D''))$

.3 $(\nabla (D \sqcap D') = \nabla D \cap SID')$

.4 $(D' \subset\subset D \to D' = \bigvee \alpha \in D(D' \operatorname{sb} \alpha))$

.5 $(D' \subset\subset D \to \nabla D' \subset \nabla D)$

.6 $(D \sqcap D' \subset\subset D \wedge D \sqcap D' \subset\subset D')$

.7 $(D \in \operatorname{dsjn} \wedge D' \in \operatorname{dsjn} \to D \sqcap D' \in \operatorname{dsjn})$

.8 $(D \in \operatorname{dsjn} \wedge D' \subset\subset D \wedge \alpha \in D \wedge \alpha' \in D' \wedge \alpha\alpha' \neq 0 \to \alpha' \subset \alpha)$

.9 $(\alpha\beta = 0 \to \operatorname{sb} \alpha \operatorname{sb} \beta = 1)$

.10 $(\alpha\beta = 0 \to \sim 1 \operatorname{sb} \alpha \operatorname{sb} \beta = 0)$

.11 $(D \in \operatorname{dsjn} \wedge D' \subset \sim 1 \wedge \alpha \in D \wedge \beta \in D \wedge \alpha \neq \beta \to D' \operatorname{sb} \alpha \cap D' \operatorname{sb} \beta = 0)$

.12 $(\alpha \in D \in \operatorname{dsjn} \wedge D' \subset\subset D \to \nabla(D' \operatorname{sb} \alpha) = \nabla D'\alpha)$

.13 $(\alpha \in D \in \operatorname{dsjn} \wedge D' \subset\subset D \wedge \nabla D' = \nabla D \to \nabla(D' \operatorname{sb} \alpha) = \alpha)$

.14 $(D \in \operatorname{scheme} \varphi \wedge D' \in \operatorname{scheme} \varphi \to D \sqcap D' \in \operatorname{scheme} \varphi)$

A Preliminary Integral

With Riemann sums in mind we make

Definition

3.45 $(\operatorname{rsum} f\xi\varphi \equiv \sum \beta \in \operatorname{dmn} \xi (\boldsymbol{.}\, f \,\boldsymbol{.}\, \xi\beta \bullet \boldsymbol{.}\, \varphi\beta))$

Definition

3.46 $(\operatorname{selector} \equiv \mathsf{E}\, \xi \in \operatorname{To} \mathrm{U} \bigwedge \beta \in \operatorname{dmn} \xi (\boldsymbol{.}\, \xi\beta \in \beta))$

Definition

3.47 $(\operatorname{Mode} \varphi \equiv \mathsf{E}\, D, \xi (D \in \operatorname{scheme} \varphi \wedge \xi \in \operatorname{selector} \wedge \operatorname{dmn} \xi \in \operatorname{scheme} \varphi \wedge \operatorname{dmn} \xi \subset\subset D))$

3.48 Definitions

.0 $(⨍ f\varphi\} \equiv \operatorname{lm} \xi \operatorname{Mode} \varphi \operatorname{rsum} f\xi\varphi)$

.1 $(\overline{\int} f\varphi\} \equiv \overline{\operatorname{lm}}\, \xi \operatorname{Mode} \varphi \operatorname{rsum} f\xi\varphi)$

.2 $(\underline{\int} f\varphi\} \equiv \underline{\operatorname{lm}}\, \xi \operatorname{Mode} \varphi \operatorname{rsum} f\xi\varphi)$

Definition

3.49 $(\int \# \, \underline{\mathrm{u}} x \varphi \, \mathrm{d}\, x \equiv ⨍ \curlywedge x \underline{\mathrm{u}} x \varphi\})$

Definition

3.50 $(\operatorname{ladder} f\lambda \equiv (\operatorname{sng}' \mathsf{E}\, x(\boldsymbol{.}\, fx = 0) \cup \bigvee n \in \omega' \operatorname{sng}' \mathsf{E}\, x(^{\bullet}\lambda n \leq \boldsymbol{.}\, fx < {}^{\bullet}\lambda(n+1)) \cup \operatorname{sng}' \mathsf{E}\, x(\boldsymbol{.}\, fx = \infty)))$

The central integral in 3.49 is preliminary. A theory more attractive in a number of respects emerges if the integral in 3.49 is narrowed somewhat in the infinite cases. This will be done in 3.68.

Theorem

3.51 $(\varphi \in \mathrm{Ms} \to \operatorname{dmn} \operatorname{Mode} \varphi = \operatorname{scheme} \varphi \wedge \operatorname{run} \operatorname{is} \operatorname{Mode} \varphi)$

Notice that the principle of choice is involved.

3.52 Lemmas

.0 $(\beta \in \operatorname{dmn}\xi \leftrightarrow .\,\xi\beta \in \mathrm{U} \leftrightarrow .\,\xi\beta \neq \mathrm{U})$

.1 $(g = \curlywedge x\,.\,fx \rightarrow \operatorname{dmn} g = \operatorname{dmn} f \wedge .\,gx = .\,fx)$

.2 $(f = \curlywedge x\underline{\mathrm{u}}x \wedge g = \curlywedge x\underline{\mathrm{v}}x \wedge \underline{\mathrm{u}}x = \underline{\mathrm{v}}x \rightarrow .\,fx = .\,gx)$

.3 $(f = \curlywedge x\underline{\mathrm{u}}x \rightarrow \bigwedge x \in \mathrm{U}(.\,fx \in A \leftrightarrow \underline{\mathrm{u}}x \in A))$

.4 $(f = \curlywedge x\underline{\mathrm{u}}x \wedge g = \curlywedge x\underline{\mathrm{v}}x \wedge h = \curlywedge x(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) \rightarrow .\,hx = .\,fx + .\,gx)$

.5 $(f = \curlywedge x\underline{\mathrm{u}}x \wedge g = \curlywedge x(c \cdot \underline{\mathrm{u}}x) \rightarrow .\,gx = c \cdot .\,fx)$

Theorem

3.53 $(\int\# \;.\,fx\varphi \,\mathrm{d}\,x = ⨍ f\varphi\})$

Theorem

3.54 $(\operatorname{Alm}\varphi x(.\,fx = .\,gx) \rightarrow ⨍ f\varphi\} = ⨍ g\varphi\})$

Proof:

Let

$(M = \operatorname{Mode}\varphi \wedge A = \sqsubseteq x \in \operatorname{rlm}\varphi(.\,fx = .\,gx) \wedge B = \sqsubseteq x \in \operatorname{rlm}\varphi(.\,fx \neq .\,gx))$

We know of course that

$(.\,\varphi B = 0)$

Now let

$(D = \operatorname{sng}' A \cup \operatorname{sng}' B)$.

Note that

$(D \in \operatorname{scheme}\varphi)$

and that

$(\xi \in \operatorname{vs} MD \wedge \beta \in \operatorname{dmn}\xi$
$\rightarrow \beta \subset A \vee \beta \subset B$
$\rightarrow .\,f\,.\,\xi\beta \bullet .\,\varphi\beta = .\,g\,.\,\xi\beta \bullet .\,\varphi\beta)$.

Thus

$(\xi \in \operatorname{vs} MD \rightarrow$
$\operatorname{rsum} f\xi\varphi$
$= \sum \beta \in \operatorname{dmn}\xi(.\,f\,.\,\xi\beta \bullet .\,\varphi\beta)$
$= \sum \beta \in \operatorname{dmn}\xi(.\,g\,.\,\xi\beta \bullet .\,\varphi\beta)$
$= \operatorname{rsum} g\xi\varphi)$.

Accordingly,

$(⨍ f\varphi\} = ⨍ g\varphi\})$.

We now have almost at once

Theorem

3.55 $(\operatorname{Alm}\varphi x(\underline{\mathrm{u}}x = \underline{\mathrm{v}}x) \rightarrow \int\# \,\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x = \int\# \,\underline{\mathrm{v}}x\varphi \,\mathrm{d}\,x)$

3.56 Theorems

.0 $(J = \int\# \,\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x + \int\# \,\underline{\mathrm{v}}x\varphi \,\mathrm{d}\,x \in \mathrm{rl} \rightarrow J = \int\# \,(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x)\varphi \,\mathrm{d}\,x)$

.1 $(0 \neq c \in \mathrm{rf} \rightarrow c \cdot \int\# \,\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x = \int\# \,(c \cdot \underline{\mathrm{u}}x)\varphi \,\mathrm{d}\,x)$

.2 $(\varphi \in \mathrm{Ms} \rightarrow \int\# \,0\varphi \,\mathrm{d}\,x = 0)$

.3 $(\varphi \in \operatorname{Msr} S \rightarrow \int\# \,1\varphi \,\mathrm{d}\,x = .\,\varphi S)$

.4 $(A \in \operatorname{mbl}\varphi \rightarrow \int\# \;\operatorname{Cr} xA\varphi \,\mathrm{d}\,x = .\,\varphi A)$

.5 $(\varphi \in \mathrm{Ms} \wedge K \in \mathrm{fnt} \wedge J = \sum n \in K \int\# \,\underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\,x \in \mathrm{rl} \rightarrow J = \int\# \;\sum n \in K\underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\,x)$

.6 $(\operatorname{Alm}\varphi x(\underline{\mathrm{u}}x \geq 0) \rightarrow \int\# \,\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x\backsim < 0)$

.7 $(\operatorname{Alm}\varphi x(\underline{\mathrm{u}}x \leq \underline{\mathrm{v}}x) \rightarrow \int\# \,\underline{\mathrm{v}}x\varphi \,\mathrm{d}\,x\backsim < \int\# \,\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x)$

Theorem
3.57 $(\int\# \underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x \in \mathrm{rl} \to \mathrm{Alm}\,\varphi x(\underline{\mathrm{u}}x \in \mathrm{rl}))$

Proof:

Let

$(M = \mathrm{Mode}\,\varphi \wedge S = \mathrm{rlm}\,\varphi \wedge A = \sqsubseteq x \in S(\underline{\mathrm{u}}x \backsim\in \mathrm{rl}) \wedge f = \curlywedge x\underline{\mathrm{u}}x)$

and notice that

$\bigwedge x \in A(\centerdot\, fx \backsim\in \mathrm{rl})$.

Now we choose

$(D \in \mathrm{scheme}\,\varphi)$

that

$\bigwedge \xi \in \mathrm{vs}\,MD(\sum \beta \in \mathrm{dmn}\,\xi(\centerdot\, f \centerdot\, \xi\beta \bullet \centerdot\, \varphi\beta) \in \mathrm{rl})$

and let

$(D' = \sqsubseteq \beta \in D(\beta A \neq 0))$

and so choose

$(\xi' \in \mathrm{On}\,D\,\mathrm{selector})$

that

$\bigwedge \beta \in D'(\centerdot\, \xi'\beta \in \beta A)$.

Since evidently

$(\sum \beta \in D'(\centerdot\, f \centerdot\, \xi'\beta \bullet \centerdot\, \varphi\beta) \in \mathrm{rl})$

we are sure

$\bigwedge \beta \in D'(\centerdot\, f \centerdot\, \xi'\beta \bullet \centerdot\, \varphi\beta \in \mathrm{rl} \wedge \centerdot\, f \centerdot\, \xi'\beta \backsim\in \mathrm{rl} \wedge \centerdot\, \varphi\beta = 0)$.

Consequently

$(A \subset \nabla D' \wedge 0 \leq \centerdot\, \varphi A \leq \centerdot\, \varphi \nabla D' = 0 \wedge \mathrm{Alm}\,\varphi x(\underline{\mathrm{u}}x \in \mathrm{rl}))$.

Replacing ‘rl’ by ‘rf’ in the above argument yields

Theorem
3.58 $(\int\# \underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x \in \mathrm{rf} \to \mathrm{Alm}\,\varphi x(\underline{\mathrm{u}}x \in \mathrm{rf}))$

Lemma
3.59 $(f \in \mathrm{ff{+}}\,\mathrm{mbl}\,\varphi \wedge 1 < \lambda < \infty \to \mathrm{ladder}\,f\lambda \in \mathrm{scheme}\,\varphi)$

Lemma
3.60 $(f \in \mathrm{ff{+}}\,\mathrm{mbl}\,\varphi \wedge 1 < \lambda < \infty \wedge M = \mathrm{Mode}\,\varphi \wedge D = \mathrm{ladder}\,f\lambda \wedge \xi \in \mathrm{vs}\,MD \wedge$
$G = \curlywedge \alpha \inf x \in \alpha \centerdot\, fx \wedge S = \sum \alpha \in D(\centerdot\, G\alpha \bullet \centerdot\, \varphi\alpha)$
$\to 0 \leq S \leq \mathrm{rsum}\,f\xi\varphi \leq \lambda \cdot S)$

Proof:

Let

$(D' = \mathrm{dmn}\,\xi)$

and check first that

$(x \in \alpha \in D \to \centerdot\, G\alpha \leq \centerdot\, fx \leq \lambda \cdot \centerdot\, G\alpha)$.

Since

$(D' \in \mathrm{scheme}\,\varphi \wedge D' \subset\subset D)$

we may now use 3.44.13, 1.24.3, 3.44.11, 1.39, and 3.44.4 in checking

$(0 \leq S$
$= \sum \alpha \in D(\centerdot G\alpha \bullet \centerdot \varphi\alpha)$
$= \sum \alpha \in D(\centerdot G\alpha \bullet \centerdot \varphi \nabla(D' \operatorname{sb} \alpha))$
$= \sum \alpha \in D(\centerdot G\alpha \bullet \sum \beta \in D' \cap \operatorname{sb} \alpha \centerdot \varphi\beta)$
$= \sum \alpha \in D \sum \beta \in D' \cap \operatorname{sb} \alpha(\centerdot G\alpha \bullet \centerdot \varphi\beta)$
$= \sum \alpha \in D \sum \beta \in D' \cap \operatorname{sb} \alpha(\centerdot f \centerdot \xi\beta \bullet \centerdot \varphi\beta)$
$= \sum \beta \in \bigvee \alpha \in D(D' \cap \operatorname{sb} \alpha)(\centerdot f \centerdot \xi\beta \bullet \centerdot \varphi\beta)$
$= \sum \beta \in D'(\centerdot f \centerdot \xi\beta \bullet \centerdot \varphi\beta)$
$= \operatorname{rsum} f\xi\varphi$
$= \sum \alpha \in D \sum \beta \in D' \cap \operatorname{sb} \alpha(\centerdot f \centerdot \xi\beta \bullet \centerdot \varphi\beta)$
$\leq \sum \alpha \in D \sum \beta \in D' \cap \operatorname{sb} \alpha((\lambda \cdot \centerdot G\alpha) \bullet \centerdot \varphi\beta)$
$= \lambda \cdot \sum \alpha \in D \sum \beta \in D' \cap \operatorname{sb} \alpha(\centerdot G\alpha \bullet \centerdot \varphi\beta)$
$= \lambda \cdot S)$

The proof is complete.

From 3.60 we easily infer

Lemma

3.61 $(f \in \mathrm{ff}{+} \operatorname{mbl} \varphi \wedge 1 < \lambda < \infty \wedge D = \operatorname{ladder} f\lambda \wedge G = \curlywedge \alpha \inf x \in \alpha \centerdot fx \wedge S = \sum \alpha \in D(\centerdot G\alpha \bullet \centerdot \varphi\alpha)$
$\rightarrow 0 \leq \overline{\int} f\varphi\} \leq \lambda \cdot S \leq \lambda \cdot \underline{\int} f\varphi\})$

We now see at once

Theorem

3.62 $(f \in \mathrm{ff}{+} \operatorname{mbl} \varphi \rightarrow 0 \leq \int\# \, f\varphi\})$

Theorem

3.63 $(\operatorname{massable} \varphi x \underline{\mathrm{u}} x \wedge \operatorname{Alm} \varphi x(\underline{\mathrm{u}} x \geq 0) \rightarrow 0 \leq \int\# \, \underline{\mathrm{u}} x \varphi \, \mathrm{d}\, x)$

Proof:

Let

$(f = \curlywedge x \in \operatorname{rlm} \varphi \operatorname{ps} \underline{\mathrm{u}} x)$.

Note that

$(\operatorname{Alm} \varphi x(\centerdot fx = \underline{\mathrm{u}} x) \wedge f \in \mathrm{ff}{+} \operatorname{mbl} \varphi \wedge \int\# \, \underline{\mathrm{u}} x \varphi \, \mathrm{d}\, x = \int\# \, \centerdot fx\varphi \, \mathrm{d}\, x = ⨍ f\varphi\} \geq 0)$

Lemma

3.64 $(\operatorname{massable} \varphi x \underline{\mathrm{u}} x \wedge \int\# \, \underline{\mathrm{u}} x \varphi \, \mathrm{d}\, x < \infty \rightarrow \int\# \, \operatorname{ps} \underline{\mathrm{u}} x \varphi \, \mathrm{d}\, x < \infty)$

Proof:

Let

$(S = \operatorname{rlm} \varphi \wedge f = \curlywedge x \in S \underline{\mathrm{u}} x \wedge P = \curlywedge x \in S \operatorname{ps} \centerdot fx)$

and note that

$(P \in \mathrm{ff}{+} \operatorname{mbl} \varphi)$

and

.0 $\bigwedge x(\centerdot fx = \underline{\mathrm{u}} x \wedge \centerdot Px = \operatorname{ps} \centerdot fx = \operatorname{ps} \underline{\mathrm{u}} x)$.

Consequently

$(⨍ f\varphi\} = \int\# \, \centerdot fx\varphi \, \mathrm{d}\, x = \int\# \, \underline{\mathrm{u}} x \varphi \, \mathrm{d}\, x < \infty \wedge \int\# \, \operatorname{ps} \underline{\mathrm{u}} x \varphi \, \mathrm{d}\, x = ⨍ P\varphi\})$.

Let

$(D = \operatorname{ladder} P2 \wedge M = \operatorname{Mode} \varphi)$

and so determine

$(\xi \in \operatorname{vs} MD)$

that

.1 $(\sum \beta \in \operatorname{dmn} \xi(\centerdot f \centerdot \xi\beta \bullet \centerdot \varphi\beta) < \infty)$.

Next let

$(G = \curlywedge \alpha \inf x \in \alpha \,.\, Px \wedge J = \sum \alpha \in D(.\, G\alpha \bullet .\, \varphi\alpha))$.

Now from 3.60, .0, 1.22.13, .1, and 1.2.6 we see that

$$
\begin{aligned}
(0 \le J& \\
&\le \sum \beta \in \operatorname{dmn} \xi(.\, P \,.\, \xi\beta \bullet .\, \varphi\beta) \\
&= \sum \beta \in \operatorname{dmn} \xi(\operatorname{ps} .\, f \,.\, \xi\beta \bullet .\, \varphi\beta) \\
&= \sum \beta \in \operatorname{dmn} \xi \operatorname{ps}(.\, f \,.\, \xi\beta \bullet .\, \varphi\beta) \\
&< \infty) \ .
\end{aligned}
$$

Consequently we know

$(J < \infty)$

and from .0, 3.62, and 3.61 we conclude

$(\int\#\ \operatorname{ps} \underline{\mathrm{u}} x\varphi \,\mathrm{d}\,x = \int\#\ .\, Px\varphi \,\mathrm{d}\,x = ⨏ P\varphi\} = \overline{\int} P\varphi\} \le 2 \cdot J < \infty)$.

Theorem

3.65 $(\operatorname{massable} \varphi x \underline{\mathrm{u}} x \rightarrow \int\#\ \underline{\mathrm{u}} x\varphi \,\mathrm{d}\,x = \int\#\ \operatorname{ps} \underline{\mathrm{u}} x\varphi \,\mathrm{d}\,x — \int\#\ \operatorname{ng} \underline{\mathrm{u}} x\varphi \,\mathrm{d}\,x)$

Lemma

3.66 $(A \in \operatorname{mbl} \varphi \wedge y \in \operatorname{rl} \wedge P \in \operatorname{sqnc} \operatorname{ff+} \operatorname{mbl} \varphi \wedge \bigwedge x \in A(\underline{\operatorname{lin}}\, n \,..\, Pnx \ge y)$
$\rightarrow \underline{\operatorname{lin}}\, n \int\#\ (..\, Pnx \bullet \operatorname{Cr} xA)\varphi \,\mathrm{d}\,x \ge y \bullet .\, \varphi A)$

Proof:

Since the desired conclusion is evident in the event

$(y \le 0)$

we assume

$(y > 0)$.

Let

$(S = \curlywedge n \in \omega \bigwedge \nu \in \omega \backsim n \,\mathsf{E}\, x \in A(..\, Pnx > \lambda))$.

Since

$$
\begin{aligned}
(S \in \operatorname{sqnc} \subset \operatorname{mbl} \varphi \wedge A = \bigvee n \in \omega \,.\, Sn \wedge& \\
&\bigwedge n \in \omega \bigwedge x \in \operatorname{rlm} \varphi(..\, Pnx \bullet \operatorname{Cr} xA \ge \lambda \bullet \operatorname{Cr} x \,.\, Sn))
\end{aligned}
$$

we infer

$$
\begin{aligned}
(\underline{\operatorname{lin}}\, n \int\#\ (..\, Pnx \bullet \operatorname{Cr} xA)\varphi \,\mathrm{d}\,x& \\
&\ge \underline{\operatorname{lin}}\, n \int\#\ (..\, Pnx \bullet \operatorname{Cr} x \,.\, Sn)\varphi \,\mathrm{d}\,x \\
&\ge \underline{\operatorname{lin}}\, n \int\#\ (\lambda \bullet \operatorname{Cr} x \,.\, Sn)\varphi \,\mathrm{d}\,x) \\
&= \underline{\operatorname{lin}}\, n(\lambda \bullet \int\#\ \operatorname{Cr} x \,.\, Sn\varphi \,\mathrm{d}\,x) \\
&= \underline{\operatorname{lin}}\, n(\lambda \bullet .\, \varphi \,.\, Sn) \\
&= \lambda \bullet .\, \varphi A)
\end{aligned}
$$

The arbitrary nature of λ assures the desired conclusion.

Theorem

3.67 $(P \in \operatorname{sqnc} \operatorname{ff+} \operatorname{mbl} \varphi \rightarrow \underline{\operatorname{lin}}\, n \int\#\ ..\, Pnx\varphi \,\mathrm{d}\,x \ge \int\#\ \underline{\operatorname{lin}}\, n \,..\, Pnx\varphi \,\mathrm{d}\,x)$

Proof:

Let

$(\xi = \curlywedge x \in \operatorname{rlm} \varphi\, \underline{\operatorname{lin}}\, n \,..\, Pnx)$

and note

$(\xi \in \operatorname{ff+} \operatorname{mbl} \varphi \wedge \bigwedge x \in \operatorname{rlm} \varphi(.\, \xi x = \underline{\operatorname{lin}}\, n \,..\, Pnx))$.

Next let

$(1 < \lambda < \infty \wedge D = \text{ladder}\,\xi\lambda \wedge G = \measuredangle\,\alpha \inf x \in \alpha\,\boldsymbol{.}\,\xi x)$

and use 3.66 in checking

$(H \in \text{fnt}\,\text{sb}\,D \rightarrow$

$0 \leq \sum \alpha \in H\, \text{Cr}\, x\alpha \leq 1 \wedge$

$\underline{\text{lin}}\, n \int\# \,\boldsymbol{..}\, Pnx\varphi\, \mathrm{d}\, x$

$\geq \underline{\text{lin}}\, n \int\# \,(\boldsymbol{..}\, Pnx \bullet \sum \alpha \in H\, \text{Cr}\, x\alpha)\varphi\, \mathrm{d}\, x$

$= \underline{\text{lin}}\, n \int\# \,\sum \alpha \in H(\boldsymbol{..}\, Pnx \bullet \text{Cr}\, x\alpha)\varphi\, \mathrm{d}\, x$

$= \underline{\text{lin}}\, n \sum \alpha \in H \int\# \,(\boldsymbol{..}\, Pnx \bullet \text{Cr}\, x\alpha)\varphi\, \mathrm{d}\, x$

$\geq \sum \alpha \in H\, \underline{\text{lin}}\, n \int\# \,(\boldsymbol{..}\, Pnx \bullet \text{Cr}\, x\alpha)\varphi\, \mathrm{d}\, x$

$\geq \sum \alpha \in H(\boldsymbol{.}\, G\alpha \bullet \boldsymbol{.}\, \varphi\alpha))$

3.68 Definitions

.0 $(\int \underline{\text{u}}x\varphi\, \mathrm{d}\, x \equiv (\text{massile}\, \varphi x \underline{\text{u}}x \rightarrow \int\#\, \underline{\text{u}}x\varphi\, \mathrm{d}\, x))$

.1 $(\int A; \underline{\text{u}}x\varphi\, \mathrm{d}\, x \equiv \int (\text{Cr}\, xA \bullet \underline{\text{u}}x)\varphi\, \mathrm{d}\, x)$

3.69 Theorems

.0 $(\text{massile}\, \varphi x \underline{\text{u}}x \rightarrow \int \underline{\text{u}}x\varphi\, \mathrm{d}\, x = \int\#\, \underline{\text{u}}x\varphi\, \mathrm{d}\, x)$

.1 $(\int \underline{\text{u}}x\varphi\, \mathrm{d}\, x \in \text{rl} \rightarrow \text{massile}\, \varphi x \underline{\text{u}}x)$

.2 $(\int \underline{\text{u}}x\varphi\, \mathrm{d}\, x \in \text{rl} \leftrightarrow \int \underline{\text{u}}x\varphi\, \mathrm{d}\, x \neq \text{U})$

We now state without proof and shall not use:

Theorem

3.70 $(\int \underline{\text{u}}x\varphi\, \mathrm{d}\, x \in \text{rf} \rightarrow \int \underline{\text{u}}\varphi\, \mathrm{d}\, x = \int\#\, \underline{\text{u}}x\varphi\, \mathrm{d}\, x)$

3.71 Theorems

.0 $(\text{Alm}\, \varphi x(\underline{\text{u}}x = \underline{\text{v}}x) \rightarrow \int \underline{\text{u}}x\varphi\, \mathrm{d}\, x = \int \underline{\text{v}}x\varphi\, \mathrm{d}\, x)$

.1 $(\int \underline{\text{u}}x\varphi\, \mathrm{d}\, x \in \text{rl} \rightarrow \text{Alm}\, \varphi x(\underline{\text{u}}x = \text{rail}\, \underline{\text{u}}x \in \text{rl}))$

.2 $(\int \underline{\text{u}}x\varphi\, \mathrm{d}\, x \in \text{rf} \rightarrow \text{Alm}\, \varphi x(\underline{\text{u}}x \in \text{rf}))$

.3 $(J = \int \underline{\text{u}}x\varphi\, \mathrm{d}\, x + \int \underline{\text{v}}x\varphi\, \mathrm{d}\, x \in \text{rl} \rightarrow J = \int (\underline{\text{u}}x + \underline{\text{v}}x)\varphi\, \mathrm{d}\, x)$

.4 $(0 \neq c \in \text{rf} \rightarrow c \cdot \int \underline{\text{u}}x\varphi\, \mathrm{d}\, x = \int (c \cdot \underline{\text{u}}x)\varphi\, \mathrm{d}\, x)$

.5 $(\varphi \in \text{Ms} \rightarrow \int 0\varphi\, \mathrm{d}\, x = 0)$

.6 $(\varphi \in \text{Ms} \wedge c \in \text{rf} \rightarrow c \bullet \int \underline{\text{u}}x\varphi\, \mathrm{d}\, x = \int (c \bullet \underline{\text{u}}x)\varphi\, \mathrm{d}\, x)$

.7 $(A \subset \text{rlm}\, \varphi \rightarrow \int \text{Cr}\, xA\varphi\, \mathrm{d}\, x = \boldsymbol{.}\, \varphi A \leftrightarrow A \in \text{mbl}''\, \varphi)$

.8 $(\varphi \in \text{Ms} \wedge K \in \text{fnt} \wedge J = \sum n \in K \int \underline{\text{u}}' xn\varphi\, \mathrm{d}\, x \in \text{rl} \rightarrow J = \int \sum n \in K \underline{\text{u}}' xn\varphi\, \mathrm{d}\, x)$

.9 $(\text{Alm}\, \varphi x(\underline{\text{u}}x \geq 0) \rightarrow \int \underline{\text{u}}x\varphi\, \mathrm{d}\, x \backsim < 0)$

.10 $(\text{Alm}\, \varphi x(\underline{\text{u}}x \leq \underline{\text{v}}x) \rightarrow \int \underline{\text{v}}x\varphi\, \mathrm{d}\, x \backsim < \int \underline{\text{u}}x\varphi\, \mathrm{d}\, x)$

.11 $(\text{massile+}\, \varphi x \underline{\text{u}}x \leftrightarrow \int \underline{\text{u}}x\varphi\, \mathrm{d}\, x \geq 0 \wedge \text{Alm}\, \varphi x(\underline{\text{u}}x \geq 0))$

.12 $(\int \underline{\text{u}}x\varphi\, \mathrm{d}\, x = \int \text{ps}\, \underline{\text{u}}x\varphi\, \mathrm{d}\, x -\!\!\!- \int \text{ng}\, \underline{\text{u}}x\varphi\, \mathrm{d}\, x)$

.13 $(\text{massile}\, \varphi x \underline{\text{u}}x \rightarrow |\int \underline{\text{u}}x\varphi\, \mathrm{d}\, x| \leq \int |\underline{\text{u}}x|\varphi\, \mathrm{d}\, x)$

.14 $(0 < r < \infty \wedge A = \mathsf{E}\, x \in \text{rlm}\, \varphi(|\underline{\text{u}}x| \geq r) \wedge \text{massile}\, \varphi x|\underline{\text{u}}x| \rightarrow \boldsymbol{.}\, \varphi A \leq (1/r) \cdot \int |\underline{\text{u}}x|\varphi\, \mathrm{d}\, x)$

.15 $(\text{Alm}\, \varphi x(\underline{\text{u}}x = 0) \leftrightarrow \int |\underline{\text{u}}x|\varphi\, \mathrm{d}\, x = 0)$

.16 $(\text{Alm}\, \varphi x(0 \leq \underline{\text{u}}x \leq \underline{\text{v}}x) \wedge \int \underline{\text{v}}x\varphi\, \mathrm{d}\, x = 0 \rightarrow \int \underline{\text{u}}x\varphi\, \mathrm{d}\, x = 0)$

.17 $(\text{massile+}\, \varphi x \underline{\text{u}}x \wedge c \in \text{rl} \rightarrow \int (c \cdot \underline{\text{u}}x)\varphi\, \mathrm{d}\, x = c \cdot \int \underline{\text{u}}x\varphi\, \mathrm{d}\, x)$

Fatou's Lemma

3.72 $(\text{big}\, n\, \text{massile+}\, \varphi x \underline{\text{u}}' xn \rightarrow \underline{\text{lin}}\, n \int \underline{\text{u}}' xn\varphi\, \mathrm{d}\, x \geq \int \underline{\text{lin}}\, n \underline{\text{u}}' xn\varphi\, \mathrm{d}\, x)$

Proof:

Let

$(P = \leftthreetimes n \in \omega \leftthreetimes x \in \operatorname{rlm}\varphi(\operatorname{ps}\underline{\mathrm{u}}'xn \wedge \text{massile+ } \varphi\underline{\mathrm{u}}'xn))$

and check that

$(P \in \operatorname{sqnc}\text{ff+}\operatorname{mbl}\varphi \wedge \operatorname{big} n \operatorname{Alm}\varphi x(\boldsymbol{..}\,Pnx = \underline{\mathrm{u}}'xn) \wedge$

$\operatorname{Alm}\varphi x(\underline{\operatorname{lin}}\, n\,\boldsymbol{..}\,Pn\underline{\mathrm{u}}'xn = \underline{\operatorname{lin}}\, n\underline{\mathrm{u}}'xn))$.

With the help of 3.67 and 3.42.1 we now conclude

$(\underline{\operatorname{lin}}\, n \int \underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\, x$

$= \underline{\operatorname{lin}}\, n \int \boldsymbol{..}\,Pnx\varphi \,\mathrm{d}\, x$

$= \underline{\operatorname{lin}}\, n \int\# \ \boldsymbol{..}\,Pnx\varphi \,\mathrm{d}\, x$

$\geq \int\# \ \underline{\operatorname{lin}}\, n\,\boldsymbol{..}\,Pnx\varphi \,\mathrm{d}\, x$

$= \int \underline{\operatorname{lin}}\,\boldsymbol{..}\,Pnx\varphi \,\mathrm{d}\, x$

$= \int \underline{\operatorname{lin}}\, n\underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\, x)$

Lebesgue's Theorem

3.73 $(\operatorname{big} n \text{ massile+ } \varphi x\,\boldsymbol{..}\,fnx \wedge \operatorname{big} n \operatorname{Alm}\varphi x(\boldsymbol{..}\,fnx \leq \boldsymbol{..}\,f(n+1)x)$

$\rightarrow 0 \leq \operatorname{lin} n \int \boldsymbol{..}\,fnx\varphi \,\mathrm{d}\, x = \int \operatorname{lin} n\,\boldsymbol{..}\,fnx\varphi \,\mathrm{d}\, x)$

Theorem

3.74 $(\varphi \in \mathrm{Ms} \wedge K \in \operatorname{cbl} \wedge \bigwedge n \in K \text{ massile+ } \varphi x\underline{\mathrm{u}}'xn \rightarrow \int \sum n \in K\underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\, x = \sum n \in K \int \underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\, x)$

3.75 Theorems

.0 $(\varphi \in \mathrm{Ms} \wedge K \in \operatorname{cbl} \wedge \bigwedge n \in K \operatorname{massile}\varphi x\underline{\mathrm{u}}'xn \wedge \sum n \in K \int \operatorname{ps}\underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\, x < \infty$

$\rightarrow \int \sum n \in K\underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\, x = \sum n \in K \int \underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\, x < \infty)$

.1 $(\varphi \in \mathrm{Ms} \wedge K \in \operatorname{cbl} \wedge \bigwedge n \in K \operatorname{massile}\varphi x\underline{\mathrm{u}}'xn \wedge \sum n \in K \int \operatorname{ng}\underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\, x < \infty$

$\rightarrow -\infty < \int \sum n \in K\underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\, x = \sum n \in K \int \underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\, x)$

Theorem

3.76 $(\varphi \in \mathrm{Ms} \wedge K \in \operatorname{cbl} \wedge \bigwedge n \in K \operatorname{massile}\varphi x\underline{\mathrm{u}}'xn \wedge \sum \int |\underline{\mathrm{u}}'xn|\varphi \,\mathrm{d}\, x < \infty$

$\rightarrow \int \sum n \in K\underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\, x = \sum n \in K \int \underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\, x \in \mathrm{rf})$

3.77 Theorems

.0 $(-\infty < \int \underline{\mathrm{u}}x\varphi \,\mathrm{d}\, x \wedge \operatorname{big} n \operatorname{massile}\varphi x\underline{\mathrm{u}}'xn \wedge \operatorname{big} n \operatorname{Alm}\varphi x(\underline{\mathrm{u}}x \leq \underline{\mathrm{u}}'xn)$

$\rightarrow \underline{\operatorname{lin}}\, n \int \underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\, x \geq \int \underline{\operatorname{lin}}\, n\underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\, x > -\infty)$

.1 $(\int \underline{\mathrm{u}}x\varphi \,\mathrm{d}\, x < \infty \wedge \operatorname{big} n \operatorname{massile}\varphi x\underline{\mathrm{u}}'xn \wedge \operatorname{big} n \operatorname{Alm}\varphi x(\underline{\mathrm{u}}x \leq \underline{\mathrm{u}}'xn)$

$\rightarrow \overline{\operatorname{lin}}\, n \int \underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\, x \leq \int \overline{\operatorname{lin}}\, n\underline{\mathrm{u}}'xn\varphi \,\mathrm{d}\, x < \infty)$

Theorem

3.78 $(\int \underline{\mathrm{u}}x\varphi \,\mathrm{d}\, x < \infty \wedge \operatorname{big} n \operatorname{massile}\varphi x\underline{\mathrm{u}}'xn \wedge \operatorname{big} n \operatorname{Alm}\varphi x(|\underline{\mathrm{u}}'xn| \leq \underline{\mathrm{u}}x) \wedge \operatorname{Alm}\varphi x(\operatorname{lin} n\underline{\mathrm{u}}'xn \in \mathrm{rl})$

$\rightarrow \operatorname{lin} n \int |\underline{\mathrm{u}}'xn - \operatorname{lin} n\underline{\mathrm{u}}'xn|\varphi \,\mathrm{d}\, x = 0)$

Theorem

3.79 $(\varphi \in \mathrm{Ms} \wedge A \subset \operatorname{rlm}\varphi \rightarrow \int A; 1\varphi \,\mathrm{d}\, x = \boldsymbol{.}\,\varphi A \leftrightarrow A \in \operatorname{mbl}''\varphi)$

Theorem

3.80 $(\int \underline{\mathrm{u}}x\varphi \,\mathrm{d}\, x \in \mathrm{rf} \wedge r \in \mathrm{rfp} \rightarrow \bigvee \delta \in \mathrm{rfp} \bigwedge A \in \operatorname{mbl}\varphi(\boldsymbol{.}\,\varphi A \leq \delta \rightarrow \int A; |\underline{\mathrm{u}}x|\varphi \,\mathrm{d}\, x \leq r))$

Proof:

Suppose

$(S = \operatorname{rlm}\varphi \wedge B = \leftthreetimes n \in \omega \mathsf{E}\, x \in S(|\underline{\mathrm{u}}x| \leq n) \wedge$

$f = \leftthreetimes n \in \omega \leftthreetimes x \in S(|\underline{\mathrm{u}}x| \bullet \operatorname{Cr} x\,\boldsymbol{.}\,Bn + n \bullet \operatorname{Cr} x{\sim}\,\boldsymbol{.}\,Bn))$

Now

$(\bigwedge x \in S(\mathrm{lin}\, n \,..\, fnx \leq |\underline{\mathrm{u}}x|) \wedge$

$\bigwedge n \in \omega \bigwedge x \in S(0 \leq \,..\, fnx \leq |\underline{\mathrm{u}}x|) \wedge$

$\bigwedge n \in \omega \bigwedge x \in S(0 \leq \,..\, fnx \leq n) \wedge$

$\bigwedge n \in \omega \bigwedge x \in S(0 \leq |\underline{\mathrm{u}}x| - \,..\, fnx))$

After 3.78 verifying

$(\mathrm{lin}\, n \int(|\underline{\mathrm{u}}x| - \,..\, fnx)\varphi\, \mathrm{d}\, x = 0)$

we so select

$(N \in \omega \sim 1)$

that

$(\int(|\underline{\mathrm{u}}x| - \,..\, fNx)\varphi\, \mathrm{d}\, x \leq r/2)$

and take

$(\delta = r/(2 \cdot N))$.

We now see

$(\int A; |\underline{\mathrm{u}}x|\varphi\, \mathrm{d}\, x$

$= \int(|\underline{\mathrm{u}}x| \bullet \mathrm{Cr}\, xA)\varphi\, \mathrm{d}\, x$

$= \int((|\underline{\mathrm{u}}x| - \,..\, fNx + \,..\, fNx) \bullet \mathrm{Cr}\, xA)\varphi\, \mathrm{d}\, x$

$= \int((|\underline{\mathrm{u}}x| - \,..\, fNx) \bullet \mathrm{Cr}\, xA)\varphi\, \mathrm{d}\, x + \int(\,..\, fNx \bullet \mathrm{Cr}\, xA)\varphi\, \mathrm{d}\, x$

$\leq \int(|\underline{\mathrm{u}}x| - \,..\, fNx)\varphi\, \mathrm{d}\, x + \int(N \cdot \mathrm{Cr}\, xA)\varphi\, \mathrm{d}\, x$

$\leq r/2 + N \bullet \int \mathrm{Cr}\, xA\varphi\, \mathrm{d}\, x$

$= r/2 + N \bullet .\varphi A$

$\leq r/2 + N \bullet \delta$

$\leq r/2 + r/2$

$= r)$

Measure and Mean Convergence

3.81 Definitions

.0 $(\mathrm{measure} > \varphi \equiv \mathsf{E}\, f \in \mathrm{sqnc\, Upon\, rlm}\, \varphi \bigwedge r \in \mathrm{rfp} \bigvee N \in \omega$

$\bigwedge m, n, \in \omega \backsim N(.\varphi \in \mathrm{Ms} \wedge .\varphi(\mathrm{rlm}\, \varphi \sim \mathsf{E}\, x(|\,..\, fmx - \,..\, fnx| \leq r)) \leq r))$

.1 $(\mathrm{mean} > \varphi \equiv \mathsf{E}\, f \in \mathrm{sqnc\, Upon\, rlm}\, \varphi \bigwedge r \in \mathrm{rfp} \bigvee N \in \omega$

$\bigwedge m, n, \in \omega \backsim N(\int |\,..\, fmx - \,..\, fnx|\varphi\, \mathrm{d}\, x \leq r))$

Note that

$(f \in \mathrm{measure} > \varphi \cup \mathrm{mean} > \varphi \rightarrow \varphi \in \mathrm{Ms})$.

Theorem

3.82 $(f \in \mathrm{measure} > \varphi \rightarrow \bigvee g \in \mathrm{sbqnc}\, f\, \mathrm{Alm}\, \varphi x(\mathrm{lin}\, n \,..\, gnx \in \mathrm{rf}))$

Proof:

Let

$(S = \mathrm{rlm}\, \varphi)$

and so choose

$(\mathcal{N} \in \mathrm{ndx})$

that

$\bigwedge m \bigwedge n \bigwedge \nu(.\mathcal{N}\nu \leq m \in \omega \wedge .\mathcal{N}\nu \leq n \in \omega \rightarrow .\varphi(S \sim \mathsf{E}\, x(|\,..\, fmx - \,..\, fnx| \leq \underline{2}\, \nu)) \leq \underline{2}\, \nu))$.

Let

$(g = f : \mathcal{N})$.

Suppose

.0 $(A = \curlywedge \nu \in \omega \,\mathsf{E}\, x(|\,..g(\nu+1)x - \,..g\nu x| \leq \underline{2}\,\nu))$

and check that

.1 $\bigwedge \nu \in \omega(\mathbf{.}\,\varphi(S\sim\mathbf{.}\,A\nu) \leq \underline{2}\,\nu)$.

Letting

.2 $(C = \bigvee \mu \in \omega \bigwedge \nu \in \omega \backsim \mu\,\mathbf{.}\,A\nu)$

we observe the desired conclusion is an immediate consequence of Steps 1 and 2 below.

Step 0

$(x \in C \wedge r > 0 \rightarrow \bigvee N \in \omega \bigwedge m, n, \in \omega \backsim N(|\,..gmx - \,..gnx| \leq r))$

Proof:

Refer to .2 and so select

$(\mu \in \omega)$

that

$(x \in \bigwedge \nu \in \omega \backsim \mu\,\mathbf{.}\,A\nu)$.

So choose

$(N \in \omega)$

that

$(N \geq \mu \wedge \underline{2}(N - 2) \leq r)$.

Use .0 in checking

$$\begin{aligned}
&(m, n, \in \omega \backsim N \\
&\quad \rightarrow x \in \bigwedge \nu \in \omega \backsim \mu\,\mathbf{.}\,A\nu \subset \bigwedge \nu \in \omega \backsim N\,\mathbf{.}\,A\nu \\
&\quad \rightarrow |\,..gmx - \,..gnx| \leq |\,..gmx - \,..gNx| + |\,..gnx - \,..gNx| \\
&\qquad \leq \sum \nu \in m \backsim N|\,..g(\nu+1)x - \,..g\nu x| + \sum \nu \in n \backsim N|\,..g(\nu+1)x - \,..g\nu x| \\
&\qquad \leq 2 \cdot \sum \nu \in \omega \backsim N|\,..g(\nu+1) - \,..g\nu x| \\
&\qquad \leq 2 \cdot \sum \nu \in \omega \backsim N\,\underline{2}\,\nu \\
&\qquad = \underline{2}(N - 2) \\
&\qquad \leq r)
\end{aligned}$$

Step 1

$(x \in C \rightarrow \operatorname{lin} n\,..gnx \in \mathrm{rf})$

Proof:

Use Step 0 and the fact that

$(\mathrm{rf} \in \mathrm{Complete}\,\underline{\mathrm{rf}})$.

Step 2

$(\mathbf{.}\,\varphi(S\sim C) = 0)$

Proof:

Using .2 and .1 we infer

$$\begin{aligned}
&(n \in \omega \\
&\quad \rightarrow S\sim C = \bigwedge \mu \in \omega \bigvee \nu \in \omega \backsim \mu(S\sim\mathbf{.}\,A\nu) \\
&\quad \rightarrow \mathbf{.}\,\varphi(S\sim C) \\
&\qquad \leq \sum \nu \in \omega \backsim n\,\mathbf{.}\,\varphi(S - \,\mathbf{.}\,A\nu) \\
&\qquad \leq \sum \nu \in \omega \backsim n\,\underline{2}\,\nu \\
&\qquad = \underline{2}(n - 1))
\end{aligned}$$

Since

$(\operatorname{lin} n\,\underline{2}(n - 1) = 0)$

we conclude

$(\mathbf{.}\,\varphi(S\sim C) = 0)$.

Remark

3.83 A knowledge of the technique displayed in the proof of 3.82 is almost indispensable to him who wishes to work in those branches of analysis which involve measure, integration, differentiation. Note, for example, the proof of Egoroff's Theorem (Saks, Theory of the Integral, Warsaw, 1937, p. 18).

Theorem

3.84 (mean $>\varphi \subset$ measure $>\varphi$)

Proof:

Let

$(S = \mathrm{rlm}\,\varphi \wedge f \in \mathrm{mean} > \varphi \wedge 0 < r < \infty)$

and choose

$(N \in \omega)$

so that

$\bigwedge m, n, \in \omega \backsim N(\int |\,{\bf..}\,fmx — {\bf..}\,fnx|\varphi\,\mathrm{d}\,x \leq r \cdot r)$.

Helped by 3.71.14 we infer

$\bigwedge m, n, \in \omega \backsim N($
$\quad {\bf.}\,\varphi(S \sim \mathsf{E}\, x(|\,{\bf..}\,fmx — {\bf..}\,fnx| \leq r))$
$\quad\quad = {\bf.}\,\varphi\, \mathsf{E}\, x \in S(|\,{\bf..}\,fmx — {\bf..}\,fnx| > r)$
$\quad\quad \leq {\bf.}\,\varphi\, \mathsf{E}\, x \in S(|\,{\bf..}\,fmx — {\bf..}\,fnx| \geq r)$
$\quad\quad \leq (1/r) \cdot \int |\,{\bf..}\,fmx — {\bf..}\,fnx|\varphi\,\mathrm{d}\,x$
$\quad\quad \leq (1/r) \cdot r \cdot r$
$\quad\quad = r)$

Consequently

$(f \in \mathrm{measure} > \varphi)$.

Theorem

3.85 $(g \in \mathrm{sbqnc}\, f \wedge \mathrm{Alm}\,\varphi x(\mathrm{lin}\, n\,{\bf..}\,gnx \in \mathrm{rf}) \rightarrow$

.0 $(\mathrm{lin}\, n \int |\,{\bf..}\,fnx — \underline{\mathrm{u}}x|\varphi\,\mathrm{d}\,x = 0 \rightarrow \mathrm{Alm}\,\varphi x(\mathrm{lin}\, n\,{\bf..}\,gnx = \underline{\mathrm{u}}x)) \wedge$

.1 $(f \in \mathrm{mean} > \varphi \wedge \mathrm{Alm}\,\varphi x(\mathrm{lin}\, n\,{\bf..}\,gnx = \underline{\mathrm{u}}x) \rightarrow \mathrm{lin}\, n \int |\,{\bf..}\,fnx — \underline{\mathrm{u}}x|\varphi\,\mathrm{d}\,x = 0))$

Proof of .0

Note that

$(\mathrm{Alm}\,\varphi x(\mathrm{lin}\, n({\bf..}\,gnx — \underline{\mathrm{u}}x) \in \mathrm{rf}) \wedge \mathrm{Alm}\,\varphi x(\mathrm{lin}\, n|\,{\bf..}\,gnx — \underline{\mathrm{u}}x| \in \mathrm{rf}))$.

Thus

$(0$
$= \mathrm{lin}\, n \int |\,{\bf..}\,fnx — \underline{\mathrm{u}}x|\varphi\,\mathrm{d}\,x$
$= \mathrm{lin}\, n \int |\,{\bf..}\,gns — \underline{\mathrm{u}}x|\varphi\,\mathrm{d}\,x$
$\geq \int \underline{\mathrm{lin}}\, n|\,{\bf..}\,gnx — \underline{\mathrm{u}}x|\varphi\,\mathrm{d}\,x$
$= \int |\,\underline{\mathrm{lin}}\, n|\,{\bf..}\,gnx — \underline{\mathrm{u}}x||\varphi\,\mathrm{d}\,x$
$= \int |\,\mathrm{lin}\, n|\,{\bf..}\,gnx — \underline{\mathrm{u}}x||\varphi\,\mathrm{d}\,x)$.

Consquently

$(\mathrm{Alm}\,\varphi x(\mathrm{lin}\, n|\,{\bf..}\,gnx — \underline{\mathrm{u}}x| = 0) \wedge \mathrm{Alm}\,\varphi x(\mathrm{lin}\, n\,{\bf..}\,gnx = \underline{\mathrm{u}}x))$.

Proof of .1

$(r \in \mathrm{rfp} \rightarrow \bigvee N \in \omega($
$\quad \bigwedge m, n, \in \omega \backsim N(\int |\,{\bf..}\,fmx — {\bf..}\,fnx|\varphi\,\mathrm{d}\,x \leq r) \wedge$
$\quad \bigwedge m, n, \in \omega \backsim N(\int |\,{\bf..}\,fmx — {\bf..}\,gnx|\varphi\,\mathrm{d}\,x \leq r) \wedge$
$\quad \bigwedge m \in \omega \backsim N(\int |\,{\bf..}\,fmx — \underline{\mathrm{u}}x|\varphi\,\mathrm{d}\,x \leq \underline{\mathrm{lin}}\, n \int |\,{\bf..}\,fmx — {\bf..}\,gnx|\varphi\,\mathrm{d}\,x \leq r)))$

$(0 = \mathrm{lin}\, m \int |\,{\bf..}\,fmx — \underline{\mathrm{u}}x|\varphi\,\mathrm{d}\,x = \mathrm{lin}\, n \int |\,{\bf..}\,fnx — \underline{\mathrm{u}}x|\varphi\,\mathrm{d}\,x)$

3.86 Theorems

.0 $(\text{big}\, n\, \text{massile}\, \varphi x \underline{\text{u}}' xn \wedge \text{lin}\, n \int |\underline{\text{u}}' xn — \underline{\text{u}} x| \varphi \, \mathrm{d}x = 0 \rightarrow \text{massile}\, \varphi x \underline{\text{u}} x)$

Hint. Let

$(f = \curlywedge n \in \omega \curlywedge x \in \text{rlm}\, \varphi \underline{\text{u}}' xn)$

and check that

$(f \in \text{mean} > \varphi)$.

Now use 3.84, 3.82 and the apply 3.85.0.

.1 $(\text{big}\, n (\int \underline{\text{u}}' xn \varphi \, \mathrm{d}x \in \text{rf}) \wedge \text{lin}\, n \int |\underline{\text{u}}' xn — \underline{\text{u}} x| \varphi \, \mathrm{d}x = 0 \rightarrow \text{lin}\, n \int \underline{\text{u}}' xn \varphi \, \mathrm{d}x = \int \underline{\text{u}} x \varphi \, \mathrm{d}x)$

Lebesgue Metrics

3.87 Definitions

.0 $(\text{rlb}\, \varphi \equiv \text{E}\, \psi \in \text{On}\, \text{rlm}\, \varphi \cap \text{To}\, \text{rf} (\int \centerdot \psi t \varphi \, \mathrm{d}t \in \text{rf}))$

.1 $(\text{reallebesguespace}\, \varphi \equiv \text{rlb}\, \varphi)$

.2 $(\underline{\text{rlb}}\, \varphi \equiv \curlywedge x, y, \in \text{rlb}\, \varphi \int | \centerdot xt — \centerdot yt| \varphi \, \mathrm{d}t)$

Theorem

3.88 $(\varphi \in \text{Ms} \wedge L = \text{rlb}\, \varphi \wedge \rho = \underline{\text{rlb}}\, \varphi \rightarrow \text{space}\, \rho = L \in \text{Complete}\, \rho)$

Proof:

After checking that

$(\rho \in \text{metric} \wedge \text{space}\, \rho = L)$

we complete the proof by verifying the statement

$(P \in \text{Cauchy}\, \rho \rightarrow \bigvee x \in L (P \in \text{cvg}\, \rho x))$.

Proof:

Since clearly

$(P \in \text{mean} > \varphi)$

we use 3.84 and 3.82 to so secure

$(g \in \text{sbqnc}\, P)$

that

$\text{Alm}\, \varphi t (\text{lin}\, n \centerdot\centerdot gnt \in \text{rf})$.

Next let

$(A = \text{E}\, t (\text{lin}\, n \centerdot\centerdot gnt \in \text{rf}) \wedge x = \curlywedge t \in \text{rlm}\, \varphi (\text{Cr}\, xA \bullet \text{lin}\, n \centerdot\centerdot gnt))$.

Clearly

$(x \in \text{On}\, \text{rlm}\, \varphi \cap \text{To}\, \text{rf} \wedge \text{Alm}\, \varphi t (\text{lin}\, n \centerdot\centerdot gnt = \centerdot xt))$.

Because of this and 3.85.1

$(\int \centerdot xt \varphi \, \mathrm{d}t \in \text{rf})$.

We now conclude from .0

$(\text{lin}\, n \centerdot \rho (\centerdot Pn, x) = \text{lin}\, n \int | \centerdot\centerdot Pnt — \centerdot xt| \varphi \, \mathrm{d}t = 0)$.

Approximations

3.89 Definitions

.0 $(\text{cps}\, F \equiv \text{E}\, \lambda \in \text{On}\, F \cap \text{gauge} (F \sim \text{zr}\, \lambda \in \text{cbl}))$

.1 $(\text{Crct}\, SF \equiv \text{E}\, F \bigvee \lambda \in \text{cps}\, F (f = \curlywedge x \in S \sum \beta \in F (\centerdot \lambda \beta \bullet \text{Cr}\, x \beta)))$

.2 $(\text{bsc}\, \varphi \equiv \text{E}\, F \subset \text{mbl}'' \varphi (\varphi \in \text{Ms} \wedge \bigwedge A \in \text{mbl}\, \varphi (\centerdot \varphi A = \centerdot \text{mss}\, \varphi\, \text{rlm}\, \varphi F A)))$

Note

$(\varphi = \mathcal{L} \wedge F = \bigvee a, b, \in \text{rf}\, \text{sng}\, \text{E}\, x (a < x < b) \rightarrow F \in \text{bsc}\, \varphi)$.

3.90 Definitions

.0 $(\text{Integrable}\,\varphi \equiv \text{E}\, f \in \text{Upon}\,\text{rlm}\,\varphi \cap \text{To}\,\text{rl}(\int .\, fx\varphi\,\text{d}\,x \in \text{rl}))$

.1 $(\text{Integrable+}\,\varphi \equiv (\text{gauge} \cap \text{On}\,\text{rlm}\,\varphi \cap \text{Integrable}\,\varphi))$

If you wish you may compare our treatment with that of W. W. Bledsoe and A. P. Morse, Product Measures, Trans. Amer. Math. Soc., vol. 79, No 1, pp. 182-187. Definition 3.89.2 is slightly at variance with 4.54 of Product Measures. Everything goes through however.

3.91 Theorems

.0 $(F \in \text{bsc}\,\varphi \wedge S = \text{rlm}\,\varphi \rightarrow \text{Crct}\,SF \subset \text{Integrable+}\,\varphi)$

.1 $(F \in \text{bsc}\,\varphi \wedge S = \text{rlm}\,\varphi \wedge r > 0 \wedge f \in \text{Integrable+}\,\varphi$
$\rightarrow \bigvee g \in \text{Crct}\,SF(\bigwedge x \in S(.\, fx \leq .\, gx) \wedge \int .\, gx\varphi\,\text{d}\,x \leq \int .\, fx\varphi\,\text{d}\,x + r))$

Lemma

3.92 $(F \in \text{bsc}\,\varphi \wedge S = \text{rlm}\,\varphi \wedge r > 0 \wedge \int \underline{\text{u}}x\varphi\,\text{d}\,x \in \text{rf} \rightarrow$
$\bigvee g, h, \in \text{Crct}\,SF(\int |\underline{\text{u}}x - .\, gx + .\, hx|\varphi\,\text{d}\,x \leq r \wedge \int(.\, gx + .\, hx)\varphi\,\text{d}\,x \leq \int |\underline{\text{u}}x|\varphi\,\text{d}\,x + r))$

Theorem

3.93 $(F \in \text{bsc}\,\varphi \wedge S = \text{rlm}\,\varphi \wedge r \in \text{rfp} \wedge f \in \text{rlb}\,\varphi$
$\rightarrow \bigvee g, h, \in \text{Crct}\,SF(\text{Alm}\,\varphi x(.\, fx = .\, gx - .\, hx) \wedge \int(.\, gx + .\, hx)\varphi\,\text{d}\,x \leq \int |.\, fx|\varphi\,\text{d}\,x + r))$

Proof:

With the aid of 3.92 inductively so determine

$(S' \in \text{sqnc}\,\text{Crct}\,SF \wedge T \in \text{sqnc}\,\text{Crct}\,SF)$

that

$\bigwedge n \in \omega$

.0 $(\int |.\, fx - \sum j \in n ..S'jx + \sum j \in n ..Tjx - ..S'nx + ..Tnx|\varphi\,\text{d}\,x \leq r \cdot \underline{2}(n+3) \wedge$

.1 $\int(..S'nx + ..Tnx)\varphi\,\text{d}\,x \leq \int |.\, fx - \sum j \in n ..S'jx + \sum j \in n ..Tjx|\varphi\,\text{d}\,x + r \cdot \underline{2}(n+3))$

Now let

$(g = \curlywedge x \in S \sum n \in \omega ..S'nx \wedge h = \curlywedge x \in S \sum n \in \omega ..Tnx)$.

It is easily checked that

$(g \in \text{Crct}\,SF \wedge h \in \text{Crct}\,SF)$.

From .0 we see that

$(n \in \omega \rightarrow$

.2 $(\int |.\, fx - \sum j \in n+1 ..S'jx + \sum j \in n+1 ..Tjx|\varphi\,\text{d}\,x \leq r \cdot \underline{2}(n+3))$.

From .1 we learn that

$(\int(..S'0x + ..T0x)\varphi\,\text{d}\,x \leq \int |.\, fx|\varphi\,\text{d}\,x + r/8)$

and from .1 and .2 we learn that

$(n \in \omega \rightarrow \int(..S'(n+1)x + ..T(n+1)x)\varphi\,\text{d}\,x \leq r \cdot \underline{2}(n+3) + r \cdot \underline{2}(n+4) \leq r \cdot \underline{2}(n+2))$.

Thus

$(\int(.\, gx + .\, hx)\varphi\,\text{d}\,x$
$= \sum n \in \omega \int(..S'nx + ..Tnx)\varphi\,\text{d}\,x$
$= \int(..S'0x + ..Tnx)\varphi\,\text{d}\,x + \sum n \in \omega(..S'(n+1)x + ..T(n+1)x)\varphi\,\text{d}\,x$
$\leq \int |.\, fx|\varphi\,\text{d}\,x + r/8 + \sum n \in \omega(r \cdot \underline{2}(n+2))$
$= \int |.\, fx|\varphi\,\text{d}\,x + r/8 + r/2$
$\leq \int |.\, fx|\varphi\,\text{d}\,x + r$
$< \infty)$

Accordingly

$\text{Alm}\,\varphi x(|\,.\,fx| + |\,.\,gx| + |\,.\,hx| < \infty)$

and from .2 and Fatou's Lemma it now follows that

$(\int |\,.\,fx —\ .\,gx + .\,hx|\varphi\,\mathrm{d}\,x = 0)$.

Consequently

$\text{Alm}\,\varphi x(.\,fx = .\,gx —\ .\,hx)$.

Theorem

3.94 $(F \in \text{bsc}\,\varphi \wedge \mathcal{S} = \text{rlm}\,\varphi \wedge \text{massile+}\ \varphi x\underline{\text{u}}x$
$\to \bigvee g, h, \in \text{Crct}\,\mathcal{S}F(\text{Alm}\,\varphi x(\underline{\text{u}}x = .\,gx —\ .\,hx) \wedge \int .\,hx\varphi\,\mathrm{d}\,x \leq 1))$

Proof:

In some way ascertain first, as can clearly be done, such an

$(S \in \text{sqnc}(\text{gauge} \cap \text{rlb}\,\varphi))$

that

$\text{Alm}\,\varphi x(\underline{\text{u}}x = \sum n \in \omega\,..\,Snx)$.

In accordance with 3.93 so choose

$(p, q, \in \text{sqnc}\,\text{Crct}\,\mathcal{S}F)$

that

$\bigwedge n \in \omega(\text{Alm}\,\varphi x(..\,Snx = ..\,pnx —\ ..\,qnx) \wedge \int(..\,pnx + ..\,qnx)\varphi\,\mathrm{d}\,x \leq \int ..\,Snx\varphi\,\mathrm{d}\,x + \underline{2}\,n)$.

Thus

$(n \in \omega \to$
$\quad \int ..\,pnx\varphi\,\mathrm{d}\,x + \int ..\,qnx\varphi\,\mathrm{d}\,x$
$\qquad \leq \int ..\,Snx\varphi\,\mathrm{d}\,x + \underline{2}\,n$
$\qquad = \int ..\,pnx\varphi\,\mathrm{d}\,x —\ \int ..\,qnx\varphi\,\mathrm{d}\,x + \underline{2}\,n \wedge$
$.0\quad \int ..\,qnx\varphi\,\mathrm{d}\,x \leq \underline{2}(n+1))$.

Now let

$(g = \curlywedge x \in \mathcal{S} \sum n \in \omega\,..\,pnx \wedge h = \curlywedge x \in \mathcal{S} \sum n \in \omega\,..\,qnx)$

and note

$(g, h, \in \text{Crct}\,\mathcal{S}F)$.

Because of .0

$(\int .\,hx\varphi\,\mathrm{d}\,x \leq \sum n \in \omega\,\underline{2}(n+1) = 1)$.

Accordingly

$\text{Alm}\,\varphi x(0 \leq .\,hx < \infty \wedge$
$\quad -\infty < .\,gx —\ .\,hx$
$\qquad = \sum n \in \omega\,..\,pnx —\ \sum n \in \omega\,..\,qnx$
$\qquad = \sum n \in \omega(..\,pnx —\ ..\,qnx)$
$\qquad = \sum n \in \omega\,..\,Snx$
$\qquad = \underline{\text{u}}x)$.

It is quite possible that $(F \in \text{bsc}\,\varphi)$ and yet for the members of F to be so simple in structure that it is comparatively easy to acquire information about members of $\text{Crct}\,\text{rlm}\,\varphi F$. This information can sometimes then be converted, by means of 3.95, into interesting knowledge about members of $\text{Integrable}\,\varphi$.

The reader may wish to verify Lusin's Theorem

$(\varphi \in \text{Mh}\,\rho \wedge f \in \text{ff}\,\text{mbl}\,\varphi \cap \text{To}\,\text{rf} \wedge A \in \text{mbl}'\,\varphi \cap \text{sb}\,\text{dmn}\,f$
$\to \bigwedge r \in \text{rfp} \bigvee g \in \text{Continuous}\,\rho\,\underline{\text{rf}} \bigvee C \in \text{closed}\,\rho \cap \text{sb}\,A(.\,\varphi(A{\sim}C) \leq r \wedge \bigwedge x \in C(.\,gx \leq .\,fx)))$

Theorem
3.96 $(\varphi, \psi, \in \mathrm{Msr}\, S \wedge F \in \mathrm{bsc}\, \psi \cap \mathrm{sb}\, \mathrm{mbl}\, \varphi \wedge \bigwedge \beta \in F(\,\mathbf{.}\,\psi\beta = \mathbf{.}\,\varphi\beta) \wedge \int \underline{\mathrm{u}} x\psi \,\mathrm{d}\, x \in \mathrm{rl}$
$\rightarrow \int \underline{\mathrm{u}} x\varphi \,\mathrm{d}\, x = \int \underline{\mathrm{u}} x\psi \,\mathrm{d}\, x)$

Theorem
3.97 $(\varphi \in \mathrm{Ms} \wedge \psi = \mathrm{om}\, \varphi \rightarrow \int \underline{\mathrm{u}} x\varphi \,\mathrm{d}\, x = \int \underline{\mathrm{u}} x\psi \,\mathrm{d}\, x)$

Special Integrals

3.98 Definitions
.0 $(\int ab\underline{\mathrm{u}} x\varphi \,\mathrm{d}\, x \equiv (a \leq b \wedge \int \mathrm{nt}\, ab; \underline{\mathrm{u}} x\varphi \,\mathrm{d}\, x \vee b < a \wedge -\int \mathrm{nt}\, ab; \underline{\mathrm{u}} x\varphi \,\mathrm{d}\, x \vee \sim(a, b, \in \mathrm{rl})))$
.1 $(\int \underline{\mathrm{u}} x \,\mathrm{d}\, x \equiv \int \underline{\mathrm{u}} x\, \mathcal{L} \,\mathrm{d}\, x)$
.2 $(\int A; \underline{\mathrm{u}} x \,\mathrm{d}\, x \equiv \int A; \underline{\mathrm{u}} x\, \mathcal{L} \,\mathrm{d}\, x)$
.3 $(\int ab\underline{\mathrm{u}} x \,\mathrm{d}\, x \equiv \int ab\underline{\mathrm{u}} x\, \mathcal{L} \,\mathrm{d}\, x)$

3.99 Theorems
.0 $(\sim(a, b, \in \mathrm{rl} \rightarrow \int ab\underline{\mathrm{u}} x\varphi \,\mathrm{d}\, x = \mathrm{U})$
.1 $(a < b \rightarrow \int ba\underline{\mathrm{u}} x\varphi \,\mathrm{d}\, x = -\int ab\underline{\mathrm{u}} x\varphi \,\mathrm{d}\, x)$
.2 $(\,\mathbf{.}\,\varphi\{abc\} = 0 \wedge S = \int ab\underline{\mathrm{u}} x\varphi \,\mathrm{d}\, x + \int \mathrm{nt}\, ab\underline{\mathrm{v}} x\varphi \,\mathrm{d}\, x \in \mathrm{rf} \rightarrow S = \int ac\underline{\mathrm{u}} x\varphi \,\mathrm{d}\, x)$

Chapter 4: Completely Additive Functions

Definition

As usually defined the sum of two such may not be one. This is an effect that we purpose to remedy.

4.0 $(\nabla\mathrm{meet} \equiv \mathsf{E}\, F \in {\sim}1 \bigwedge A \in F(\bigvee\beta \in F\,\mathrm{sng}(A\beta) = F\,\mathrm{sb}\,A \in \nabla\mathrm{field}))$

4.1 Definitions

- .0 $(\mathrm{grator} \equiv \mathsf{E}\,\varphi \in \mathrm{To}\,\mathrm{rl}(.\,\varphi 0 = 0))$
- .1 $(\mathrm{additive}''\,\varphi \equiv \mathsf{E}\,A \in \mathrm{dmn}\,\varphi \wedge \mathrm{function\,is}\,\varphi$
  $\bigwedge B \in \mathrm{dmn}\,\varphi \bigwedge G \in \mathrm{cbl} \cap \mathrm{dsjn} \cap \mathrm{sb}\,\mathrm{dmn}\,\varphi(.\,\varphi(A{\sim}B \,\nabla\, G) = \sum\beta \in G\,.\,\varphi(A{\sim}B\beta) \in \mathrm{rl}))$
- .2 $(\mathrm{addor} \equiv \mathsf{E}\,\varphi \in \mathrm{grator}(\mathrm{dmn}\,\varphi \subset \mathrm{additive}''\,\varphi))$
- .3 $(\mathrm{addor}'' \equiv \mathsf{E}\,\varphi \in \mathrm{addor}(\mathrm{rlm}\,\varphi \in \nabla''\,\mathrm{dmn}\,\varphi))$
- .4 $(\mathrm{dor}{+}\,\varphi \equiv \curlywedge\beta \in \mathrm{dmn}\,\varphi\,\mathrm{sup}\,\alpha \in \mathrm{dmn}\,\varphi \cap \mathrm{sb}\,\beta\,.\,\varphi\alpha)$
- .5 $(\mathrm{dor}{-}\,\varphi \equiv \mathrm{dor}{+}{-}\varphi)$
- .6 $(\mathrm{dor}{\pm}\,\varphi \equiv (\mathrm{dor}{+}\,\varphi - \mathrm{dor}{-}\,\varphi))$
- .7 $(\mathrm{vrn}\,\varphi \equiv (\mathrm{dor}{+}\,\varphi + \mathrm{dor}{-}\,\varphi))$
- .8 $(\mathrm{var}\,\varphi \equiv \mathrm{mr}\,\mathrm{vrn}\,\varphi)$
- .9 $(\mathrm{Gr}\,\mu\varphi \equiv \curlywedge\beta \in \mathrm{mbl}\,\varphi \int \beta;.\,\mu x\varphi\,\mathrm{d}\,x)$
- .10 $(\max\mu\nu \equiv \curlywedge x\,\mathrm{Sup}\{.\,\mu x\,.\,\nu x\})$
- .11 $(\mathrm{Hilt}\,F \equiv \mathsf{E}\,\varphi \in \mathrm{Msr}\,\nabla\,F(F \in \nabla\mathrm{meet}\,\mathrm{bsc}\,\varphi \wedge \nabla\,F \in \mathrm{mbl}''\,\varphi))$

Because of certain applications we have become interested in
4.2 Definitions

- .0 $(\mathrm{dmn}{+}\,\varphi \equiv \mathsf{E}\,\beta(.\,\varphi\beta > 0))$
- .1 $(\mathrm{paddor} \equiv \mathsf{E}\,\varphi \in \mathrm{grator}(\mathrm{dmn}\,\varphi \in \nabla\mathrm{meet} \wedge$
  $\bigvee A \in \mathrm{dmn}\,\varphi\,\mathrm{dsn}'(\mathrm{dmn}{+}\,\varphi\,\mathrm{sb}\,A) \subset \mathrm{dmn}{+}\,\varphi \subset \mathrm{additive}''\,\varphi))$
- .2 $(\mathrm{topaddor} \equiv \mathsf{E}\,\varphi \in \mathrm{paddor} \bigwedge G \in \mathrm{cbl} \cap \mathrm{dsjn} \cap \mathrm{sb}\,\mathrm{dmn}\,\varphi \bigwedge A \in \mathrm{dmn}\,\varphi(\sum\beta \in G\,.\,\varphi(A\beta) < \infty))$

4.3 Theorems

- .0 $(T \in \mathrm{additive}''\,\varphi \rightarrow .\,\varphi 0 = 0)$
- .1 $(T \in \mathrm{additive}''\,\varphi \wedge G \in \mathrm{cbl}\,\mathrm{dsjn}\,\mathrm{sb}\,\mathrm{dmn}\,\varphi \rightarrow .\,\varphi(T \,\nabla\, G) = \sum\beta \in G\,.\,\varphi(T\beta) \in \mathrm{rl})$
- .2 $(T \in \mathrm{additive}''\,\varphi \wedge A \in \mathrm{dmn}\,\varphi \rightarrow .\,\varphi T = .\,\varphi(TA) + .\,\varphi(T{\sim}A) \in \mathrm{rl})$
- .3 $(T \in \mathrm{additive}''\,\varphi \wedge A \in \mathrm{dmn}\,\varphi \wedge B \in \mathrm{dmn}\,\varphi \rightarrow .\,\varphi(TA \cup TB) + .\,\varphi(TAB) = .\,\varphi(TA) + .\,\varphi(TB))$

Theorem
4.4 $(\varphi \in \mathrm{addor} \leftrightarrow \varphi \in \mathrm{add}'' \wedge \mathrm{dmn}\,\varphi \in \nabla\mathrm{meet})$

Theorem
4.5 $(c \in \mathrm{rf}\,{\sim}1 \wedge \varphi \in \mathrm{addor} \rightarrow c \cdot \varphi \in \mathrm{addor}\,\mathrm{On}\,\mathrm{rlm}\,\varphi)$

Lemma
4.6 $(F_1 \in \nabla\mathrm{meet} \wedge F_2 \in \nabla\mathrm{meet} \wedge A \in F_1 \cap F_2 \rightarrow \bigvee\beta \in F_1 \cap F_2\,\mathrm{sng}(A\beta) = F_1 \cap F_2 \cap \mathrm{sb}\,A)$

Of real interest to us is

Theorem

4.7 $(\varphi_1 \in \text{addor} \wedge \varphi_2 \in \text{addor} \wedge \varphi = \varphi_1 + \varphi_2 \rightarrow$

.0 $\bigwedge A \in \text{dmn}\,\varphi(\text{dmn}\,\varphi\,\text{sb}\,A = \text{dmn}\,\varphi_1 \cap \text{dmn}\,\varphi_2 \cap \text{sb}\,A) \wedge$

.1 $\varphi \in \text{addor})$

Theorem

4.8 $(c \in \text{rf} \wedge \varphi \in \text{addor} \rightarrow c \cdot \varphi \in \text{addor})$

4.9 Theorems

.0 $(A \in \text{additive}''\,\varphi \rightarrow \text{strc}\,\varphi\,\text{sb}\,A \in \text{addor})$

.1 $(\varphi \in \text{addor} \wedge G \in \nabla\text{meet}\,\text{sb}\,\text{dmn}'\,\varphi \rightarrow \text{strc}\,\varphi G \in \text{addor})$

.2 $(\varphi \in \text{addor} \rightarrow \text{strc}\,\varphi\,\text{dmn}'\,\varphi \in \text{addor})$

4.10 Lemmas

.0 $(\varphi \in \text{grator} \rightarrow \text{dor}{+}\,\varphi \in \text{grator}\,\text{gauge}\,\text{On}\,\text{dmn}\,\varphi)$

.1 $(\varphi \in \text{paddor} \wedge \theta = \text{dor}{+}\,\varphi \wedge G \in \text{cbl}\,\text{dsjn}\,\text{sb}\,\text{dmn}\,\varphi \wedge \nabla G = A \in \text{dmn}\,\varphi$

$\rightarrow 0 \leq .\theta A = \sum \beta \in G\,.\theta\beta)$

Proof:

On the one hand

$(.\theta A > \sum \beta \in G\,.\theta\beta \rightarrow \bigvee \alpha \in \text{dmn}\,\varphi \cap \text{sb}\,A$

$(.\varphi\alpha > \sum \beta \in G\,.\theta\beta \geq 0 \wedge$

$0 < .\varphi\alpha = .\varphi(\alpha \nabla G) = \sum \beta \in G\,.\varphi(\alpha\beta) \leq \sum \beta \in G\,.\theta\beta < .\varphi\alpha \wedge$

$.\varphi\alpha < .\varphi\alpha))$

On the other hand

$(.\theta A < \sum \beta \in G\,.\theta\beta \rightarrow \bigvee H \in \text{fnt} \cap \text{sb}\,G$

$(.\theta A < \sum \beta \in H\,.\theta\beta \wedge H \neq 0 \wedge \bigwedge \beta \in H(.\theta\beta > 0) \wedge$

$\bigvee \xi$

$(\bigwedge \beta \in H(.\xi\beta \in \text{dmn}{+}\,\varphi \cap \text{sb}\,\beta) \wedge$

$\sum \beta \in H\,.\varphi\,.\xi\beta$

$> .\theta A$

$\geq .\theta \bigvee \beta \in H\,.\xi\beta$

$\geq .\varphi \bigvee \beta \in H\,.\xi\beta$

$= \sum \beta \in H\,.\varphi\,.\xi\beta \wedge$

$\sum \beta \in H\,.\varphi\,.\xi\beta > \sum \beta \in H\,.\varphi\,.\xi\beta)))$

Helped by 4.2, 4.10, and 4.4 we infer

4.11 Theorems

.0 $(\text{addor} \subset \text{paddor})$

.1 $(\varphi \in \text{paddor} \wedge \theta = \text{dor}{\pm}\,\varphi \rightarrow \theta \in \text{addor}\,\text{gauge}\,\text{On}\,\text{dmn}\,\varphi)$

Theorem

4.12 $(A \in \text{additive}''\,\varphi\,\text{dmn}'\,\varphi \rightarrow \sup \alpha \in \text{dmn}\,\varphi \cap \text{sb}\,A |\,.\varphi\alpha| < \infty)$

Proof:

Note that

$(\text{dmn}\,\varphi \cap \text{sb}\,A \subset \text{dmn}'\,\varphi)$

and let

$(F = \text{dmn}\,\varphi \cap \text{sb}\,A \wedge N = \mathsf{E}\,\beta \in F(\sup \alpha \in F \cap \text{sb}\,\beta |\,.\varphi\alpha| = \infty) \wedge M = \mathsf{E}\,\beta \in F(|\,.\varphi\beta| \geq 1))$.

After checking

$(C \in N \to \bigvee D \subset C(D \in M \wedge C{\sim}D \in MN))$

we readily infer

$(B \in F \wedge A{\sim}B \in N \to \bigvee B' \supset B(B'{\sim}B \in M \wedge A{\sim}B' \in N))$.

Because of this and 2.19

$(A \in N \to \bigvee S \in \text{sqnc}{\subset}F(\mathbf{.}S0 = 0 \wedge$
$\bigwedge n \in \omega(\mathbf{.}S(n+1){\sim}\mathbf{.}Sn \in M \wedge A{\sim}\mathbf{.}S(n+1) \in N) \wedge$
$\infty > \sum n \in \omega|\mathbf{.}S(n+1){\sim}\mathbf{.}Sn| \geq \sum n \in \omega 1 = \infty))$.

Accordingly we conclude

$(A \backsim\!\in N)$

and the desired conclusion is at hand.

Lemma

4.13 $(\varphi \in \text{topaddor} \wedge \theta = \text{dor+}\,\varphi \wedge A \in \text{dmn}\,phi \to 0 \leq \mathbf{.}\theta A < \infty)$

Proof:

Assume instead that

$(\mathbf{.}\theta A = \infty)$

and let

$(F = \mathsf{E}\,\beta \subset A(\mathbf{.}\theta\beta = 1))$.

Notice that

$(\beta \in F \to \mathbf{.}\theta(A{\sim}\beta) = \infty \to \bigvee \beta' \supset \beta(\beta'{\sim}\beta \in F \wedge \beta' \in F))$.

Choose then

$(S \in \text{sqnc}{\subset}F)$

so that

$\bigwedge n \in \omega(\mathbf{.}S(n+1){\sim}\mathbf{.}Sn \in F)$.

Now put

$(G = \bigvee n \in \omega\, \text{sng}(\mathbf{.}S(n+1){\sim}\mathbf{.}Sn))$

and use 4.2.2 and 2.18 in checking

$(\infty > \sum \beta \in G(\mathbf{.}\varphi(A\beta)$
$= \sum \beta \in G\,\mathbf{.}\varphi\beta$
$= \sum n \in \omega\,\mathbf{.}\varphi(\mathbf{.}S(n+1){\sim}\mathbf{.}Sn)$
$\geq \sum n \in \omega 1$
$= \infty)$

Helped by 4.10, 4.11, and 4.13 we infer

4.14 Theorems

.0 $(\varphi \in \text{addor Upon sb}\,A \wedge \mathbf{.}\varphi A < \infty \to \varphi \in \text{topaddor})$

.1 $(\varphi \in \text{topaddor} \wedge \theta = \text{dor+}\,\varphi \to \theta \in \text{addor gauge On dmn}\,\varphi\,\text{To rf})$

4.15 Lemmas

.0 $(\varphi \in \text{addor Upon sb}\,A \wedge \mathbf{.}phiA < \infty \wedge \psi = \text{dor}\pm\,\varphi \to \mathbf{.}\psi A = \mathbf{.}\varphi A)$

Proof:

Let

$(\varphi_1 = \text{dor+}\,\varphi \wedge \varphi_2 = \text{dor}\,\text{—}\,\varphi)$.

Because of 4.14

$(0 \leq \mathbf{.}\varphi A < \infty)$.

Also

$(\alpha \in \operatorname{dmn}\varphi \rightarrow$

$-\,.\,\varphi_2 A \leq .\,\varphi\alpha \leq \varphi_1 A < \infty \wedge$

$.\,\varphi A = .\,\varphi\alpha + .\,\varphi(A{\sim}\alpha) \geq .\,\varphi\alpha -\!\!-\; .\,\varphi_2 A \wedge$

$.\,\varphi A = .\,\varphi(A{\sim}\alpha) + .\,\varphi\alpha \leq .\,\varphi_1 A -\!\!-\; -.\,\varphi\alpha)$.

Hence

$(.\,\varphi A \geq .\,\varphi_1 A -\!\!-\; .\,\varphi_2 A \wedge .\,\varphi A \leq .\,\varphi_1 A -\!\!-\; .\,\varphi_2 A \wedge .\,\varphi A = .\,\varphi_1 A -\!\!-\; .\,\varphi_2 A = .\,\psi A)$.

Apply .0 to $-\varphi$ to learn

.1 $(\varphi \in \operatorname{addor} \operatorname{Upon} \operatorname{sb} A \wedge -\infty < .\,\varphi A \wedge \psi = \operatorname{dor}\pm\,\varphi \rightarrow .\,\psi A = .\,\varphi A)$

From .0 and .1 we learn

.2 $(\varphi \in \operatorname{addor} \operatorname{Upon} \operatorname{sb} A \wedge A \in \operatorname{dmn}\varphi \wedge \psi = \operatorname{dor}\pm\,\varphi \rightarrow .\,\psi A = .\,\varphi A)$

Theorem

4.16 $(A \in \operatorname{additive}''\,\varphi \wedge \psi = \operatorname{dor}\pm\,\varphi \rightarrow .\,\psi A = .\,\varphi A)$

Proof:

Let

$(\mu = \operatorname{strc}\varphi\operatorname{sb} A \wedge \varphi_1 = \operatorname{dor}+\varphi \wedge \varphi_2 = \operatorname{dor}-\!\!-\,\varphi \wedge \mu_1 = \operatorname{dor}+\mu \wedge \mu_2 = \operatorname{dor}-\!\!-\,\mu)$

With the help of 4.9.0 we learn

$(\mu \in \operatorname{addor} \operatorname{Upon} \operatorname{sb} A \wedge A \in \operatorname{dmn}\mu)$.

Since evidently

$(.\,\varphi A = .\,\mu A \wedge .\,\varphi_1 A -\!\!-\; .\,\mu_1 A \wedge .\,\varphi_2 A = .\,\mu_2 A)$

we may use 4.15.2 to conclude

$(.\,\varphi A = .\,\mu A = .\,\mu_1 A -\!\!-\; .\,\mu_2 A = .\,\varphi_1 A -\!\!-\; .\,\varphi_2 A = .\,\psi A)$.

We now have at once

The Jordan Decomposition Theorem

4.17 $(\varphi \in \operatorname{addor} \rightarrow \varphi = \operatorname{dor}\pm\,\varphi)$

4.18 Theorems

.0 $(\varphi \in \operatorname{gauge} \wedge \operatorname{dmn}\varphi \ni A \subset B \in \operatorname{additive}''\,\varphi \rightarrow .\,\varphi A \leq .\,\varphi B)$

.1 $(\varphi \in \operatorname{gauge} \wedge A \in \operatorname{additive}''\,\varphi \wedge F \in \operatorname{cbl}\operatorname{sb}\operatorname{dmn}\varphi \rightarrow .\,\varphi(A \triangledown F) \leq \sum \beta \in F\, .\,\varphi(A\beta))$

Proof:

Assume

$(\xi \in \operatorname{On} F \operatorname{Uto}\omega \wedge \bigwedge x \in F(\underline{u}x = \mathsf{E}\, y(.\,\xi y \in .\,xix) \wedge \underline{v}x = x{\sim} \triangledown \underline{u}x)$

and note that we may use 2.19 and .0 in checking

$(.\,\varphi(A \triangledown F)$

$= .\,\varphi(A \bigvee x \in F\underline{v}x$

$= .\,\varphi \bigvee x \in F(A\underline{v}x)$

$= \sum x \in F\, .\,\varphi(A\underline{v}x)$

$\leq \sum x \in F\, .\,\varphi(Ax)$

$= \sum \beta \in F\, .\,\varphi(A\beta))$

.2 $(\varphi \in \operatorname{addor}\operatorname{gauge} \wedge \psi = \operatorname{mr}\varphi \rightarrow \varphi \subset \psi \in \operatorname{Msr}\operatorname{rlm}\varphi \wedge \operatorname{dmn}\varphi \subset \operatorname{mbl}\psi)$

Lemma

4.19 $(\varphi \in \operatorname{addor} \wedge .\,\varphi A < \infty \wedge \theta = \operatorname{dor}+\varphi \rightarrow \bigvee P \subset A \bigwedge \alpha \in \operatorname{dmn}\varphi \cap \operatorname{sb} A(.\,\theta\alpha = .\,\varphi(\alpha P)))$

Proof:

Since according to 4.11 and 4.17

$(0 \leq .\theta A < \infty)$

we can so choose

$(B \in \operatorname{sqnc} \operatorname{sb} A)$

that

$\bigwedge n \in \omega(.\theta A \leq .\varphi . Bn + \underline{2}\, n)$.

Now let

$(C = \curlywedge n \in \omega \bigwedge m \in \omega \backsim n . Bm \wedge P = \bigvee n \in \omega . Cn)$.

Clearly

$$
\begin{aligned}
&(n \in \omega \\
&\quad \rightarrow .\theta A + \underline{2}\, n \\
&\qquad = .\theta(A\sim .Bn) + .\theta . Bn + \underline{2}\, n \\
&\qquad \geq .\theta(A\sim .Bn) + .\varphi . Bn + \underline{2}\, n \\
&\qquad \geq .\theta(A\sim .Bn) + .\theta A \\
&\quad \rightarrow .\theta(A\sim .Bn) \leq \underline{2}\, n) \ .
\end{aligned}
$$

Next because of 4.18.2

$$
\begin{aligned}
&(n \in \omega \rightarrow .\varphi(.Bn\sim .Cn) \\
&\quad \leq .\theta(.Bn\sim .Cn) \\
&\quad = .\theta(.Bn\sim \textstyle\bigwedge m \in \omega \backsim n . Bm) \\
&\quad \leq .\theta(A\sim \textstyle\bigwedge m \in \omega \backsim n . Bm) \\
&\quad = .\theta(A \textstyle\bigvee m \in \omega \backsim n\sim .Bm) \\
&\quad = .\theta \textstyle\bigvee m \in \omega \backsim n(A\sim .Bm) \\
&\quad \leq \textstyle\sum m \in \omega \backsim n . \theta(A\sim .Bm) \\
&\quad \leq \textstyle\sum m \in \omega \backsim n\, \underline{2}\, m \\
&\quad = 2 \cdot \underline{2}\, n)
\end{aligned}
$$

Accordingly

$$
\begin{aligned}
&(n \in \omega \rightarrow .\theta A \\
&\quad \leq .\varphi . Bn + \underline{2}\, n \\
&\quad = .\varphi . Cn + .\varphi(.Bn\sim .Cn) + \underline{2}\, n \\
&\quad \leq .\varphi . Cn + 2 \cdot \underline{2}\, n + \underline{2}\, n \\
&\quad = .\varphi . Cn + 3 \cdot \underline{2}\, n)
\end{aligned}
$$

Hence because of 2.21

$$
\begin{aligned}
&(.\theta A \leq \operatorname{lin} n . \varphi . Cn + 0 = .\varphi P \leq .\varphi A \wedge \\
&\quad .\theta A = .\varphi P)
\end{aligned}
$$

But

$$
\begin{aligned}
&(\alpha \in \operatorname{dmn} \varphi \operatorname{sb} A \\
&\quad \rightarrow 0 \leq .\theta\alpha - .\varphi(\alpha P) \\
&\qquad \leq .\theta\alpha - .\varphi(P\alpha) + .\theta(A\sim\alpha) - .\varphi(P\sim\alpha) \\
&\qquad = .\theta A - .\varphi P \\
&\qquad = 0 \\
&\quad \rightarrow .\theta\alpha = .\varphi(\alpha P))
\end{aligned}
$$

Theorem

4.20 $(\varphi \in \operatorname{addor} \wedge A \in \operatorname{dmn} \varphi \wedge \theta_1 = \operatorname{dor+} \varphi \wedge \theta_2 = \operatorname{dor-} \varphi$
$\rightarrow \bigvee P \subset A \bigwedge \alpha \in \operatorname{dmn} \varphi \cap \operatorname{sb} A(.\theta_1\alpha = .\varphi(\alpha P) \wedge .\theta_2\alpha = .\varphi(\alpha\sim P)))$

The Hahn Decomposition Theorem

4.21 $(\varphi \in \operatorname{addor}'' \rightarrow \bigvee P \subset \operatorname{rlm} \varphi(\operatorname{dor+} \varphi = \operatorname{sct} \varphi P \wedge \operatorname{dor-} \varphi = \operatorname{sct} \varphi\sim P))$

4.23 Theorems

.0 $(\varphi \in \mathrm{addor} \to \mathrm{vrn}\,\varphi \in \mathrm{addor})$

.1 $(\varphi \in \mathrm{addor} \to \mathrm{vrn}\,\varphi \subset \mathrm{var}\,\varphi \in \mathrm{Msr}\,\mathrm{rlm}\,\varphi \wedge \mathrm{dmn}\,\varphi \subset \mathrm{mbl}\,\mathrm{var}\,\varphi)$

We shall never use 4.24 and 4.25.

Theorem

4.24 $(\varphi \in \mathrm{addor} \wedge P \subset \mathrm{rlm}\,\varphi \wedge P' \subset \mathrm{rlm}\,\varphi \wedge A \in \mathrm{dmn}\,\varphi \wedge \theta = \mathrm{var}\,\varphi \wedge$
$\mathrm{dor}{+}\,\varphi = \mathrm{sct}\,\varphi P = \mathrm{sct}\,\varphi P' \wedge \mathrm{dor}{-}\,\varphi = \mathrm{sct}\,\varphi{\sim}P = \mathrm{sct}\,\varphi{\sim}P'$
$\to .\,\theta(AP{\sim}P' \cup AP'{\sim}P) = 0)$

Theorem

4.25 $(\varphi \in \mathrm{addor}'' \wedge P \subset \mathrm{rlm}\,\varphi \wedge P' \subset \mathrm{rlm}\,\varphi \wedge \theta = \mathrm{var}\,\varphi \wedge$
$\mathrm{dor}{+}\,\varphi = \mathrm{sct}\,\varphi P = \mathrm{sct}\,\varphi P' \wedge \mathrm{dor}{-}\,\varphi = \mathrm{sct}\,\varphi{\sim}P = \mathrm{sct}\,\varphi{\sim}P'$
$\to .\,\theta(P{\sim}P' \cup P'{\sim}P) = 0)$

Theorem

4.26 $(\varphi \in \mathrm{Ms} \wedge \psi = \mathrm{Gr}\,\mu\varphi \to \psi \in \mathrm{addor} \wedge \mathrm{zr}\,\varphi \subset \mathrm{zr}\,\psi \wedge \bigwedge A \in \mathrm{dmn}\,\psi(\mathrm{mbl}\,\varphi\,\mathrm{sb}\,A = \mathrm{dmn}\,\psi\,\mathrm{sb}\,A))$

Theorem

4.27 $(\mu \in \mathrm{ff}{+}\,G \wedge \nu \in \mathrm{ff}{+}\,G \wedge w = \max \mu\nu$
$\to w \in \mathrm{ff}{+}\,G \wedge \bigwedge x \in \nabla G(0 \leq .\,\mu x \leq .\,wx \wedge 0 \leq .\,\nu x \leq .\,wx))$

The next lemma is crucial to the remainder of the chapter.

Lemma

4.28 $(A \in \mathrm{mbl}'\,\mu\,\mathrm{mbl}'\,\varphi \wedge M = \mathrm{mbl}\,\varphi\,\mathrm{sb}\,A \wedge \mathrm{mbl}\,\varphi \subset \mathrm{mbl}\,\mu \wedge \mathrm{zr}\,\varphi \subset \mathrm{zr}\,\mu$
$\to \bigvee f \in \mathrm{ff}{+}\,\mathrm{mbl}\,\varphi(\mathrm{strc}\,\mu M = \mathrm{strc}\,\mathrm{Gr}\,f\varphi M))$

Proof:

Let

$(H = \mathsf{E}\, u \in \mathrm{ff}{+}\,\mathrm{mbl}\,\varphi \bigwedge \alpha \in M(.\,\mathrm{Gr}\,u\varphi\alpha \leq .\,\mu\alpha))$

and

.0 $(N = \sup u \in H\,.\,\mathrm{Gr}\,u\varphi\alpha)$

and note

.1 $(0 \leq N \leq .\,\mu A < \infty)$.

Since

$(u \in H \wedge v \in H \wedge w = \max uv \wedge B = \mathsf{E}\, x(.\,ux \geq .\,vx) \wedge \alpha \in M$
$\to .\,wx = \mathrm{Cr}\,xB \bullet .\,ux + \mathrm{Cr}\,x{\sim}B \bullet .\,vx \wedge$
$\int(\mathrm{Cr}\,x\alpha \bullet .\,wx)\varphi\,\mathrm{d}\,x$
$= \int(\mathrm{Cr}\,x(\alpha B) \bullet .\,ux)\varphi\,\mathrm{d}\,x + \int(\mathrm{Cr}\,x(\alpha{\sim}B) \bullet .\,vx)\varphi\,\mathrm{d}\,x$
$\leq .\,\mu(\alpha B) + .\,\mu(\alpha{\sim}B)$
$= .\,\mu\alpha$
$\to w \in H)$

it follows that

.2 $(u \in H \wedge v \in H \to \max uv \in H)$.

With the help of .0 so choose

$(p \in \operatorname{sqnc} H)$

that

$(\operatorname{lin} n \int (\operatorname{Cr} xA \bullet \,..\, pnx)\varphi \, \mathrm{d}\, x = N)$

and secure by induction such a

$(q \in \operatorname{sqnc} \mathrm{U})$

that

$(\,.\, q0 = \,.\, p0 \wedge \bigwedge n \in \omega(\,.\, q(n+1) = \max \,.\, qn \,.\, p(n+1)))$.

Evidently, because of .2 and 4.27,

$(q \in \operatorname{sqnc} H \wedge \bigwedge n \in \omega \bigwedge x \in \operatorname{rlm} \varphi(\,..\, pnx \leq \,..\, qnx \leq \,..\, q(n+1)x))$.

Now let

$(f = \not< x \in \operatorname{rlm} \varphi \operatorname{lin} n \,..\, qnx)$

and

.3 $(\mu' = \operatorname{Gr} f \varphi)$

and check that

.4 $(f \in H \wedge$

.5 $\,.\, \mu' A = N \wedge \mu' \in \operatorname{addor\,gauge} \wedge M \subset \operatorname{dmn}' \mu' \wedge$

.6 $\bigwedge \alpha \in M(\,.\, \mu'\alpha \leq \,.\, \mu\alpha < \infty) \wedge$

.7 $\operatorname{zr} \varphi \operatorname{zr} \mu')$.

It will presently be seen that

$(\alpha \in M \rightarrow \,.\, \mu'\alpha = \,.\, \mu\alpha)$.

To this end let

.8 $(\psi = \mu - \mu' \wedge$

.9 $r = 2 + 2 \cdot \,.\, \varphi A \wedge \lambda = \,.\, \psi A / r \wedge$

.10 $\psi' = \psi - \lambda \cdot \varphi \wedge \theta = \operatorname{dor+} \psi')$.

Notice that because of 4.26, 4.9.1, 4.7, 4.8, and .6

.11 $(\psi \in \operatorname{addor\,gauge} \wedge \psi' \in \operatorname{addor} \wedge A \in M \subset \operatorname{dmn}' \psi \operatorname{dmn}' \psi')$

and because of .7

.12 $(\operatorname{zr} \varphi \subset \operatorname{zr} \psi')$

and use 4.20 to secure

$(P \subset A)$

so that

.13 $\bigwedge \alpha \in \operatorname{dmn} \psi' \cap \operatorname{sb} A(\,.\, \theta\alpha = \,.\, \psi'(\alpha P))$.

Now because of .10 and .9

$$
\begin{aligned}
(\,.\, \psi' P = \,.\, \theta A & \\
& \geq \,.\, \psi A - \,.\, \psi A \cdot \,.\, \varphi A / r \\
& = \,.\, \psi A \cdot (1 - \,.\, \varphi A / r) \\
& = \,.\, \psi A \cdot ((r - \,.\, \varphi A)/r) \\
& \geq \,.\, \psi A \cdot (1 + \,.\, \varphi A)/r \\
& = \,.\, \psi A / 2)
\end{aligned}
$$

Thus

$(\,.\, \psi A \leq 2 \cdot \,.\, \psi' P)$

and because of .12

.14 $(\,.\, \varphi P = 0 \rightarrow \,.\, \psi A = 0)$.

Next let

.15 $(g = \curlywedge x \in \mathrm{rlm}\,\varphi(.\,fx + \lambda \bullet \mathrm{Cr}\,xP))$.

Note that, because of .13 and .10

.16 $(\alpha \in M \to 0 \le .\,\psi(\alpha P) — \lambda \cdot .\,\varphi(\alpha P) = .\,\psi(\alpha P) — \int(\lambda \bullet \mathrm{Cr}\,x\alpha \bullet \mathrm{Cr}\,xP)\varphi\,\mathrm{d}\,x)$

and because of .8, .3, .16, and .15

$(.\,\mu\alpha$

$= .\,\mu'\alpha + .\,\psi\alpha$

$\ge \int(\mathrm{Cr}\,x\alpha \bullet .\,fx)\varphi\,\mathrm{d}\,x + .\,\psi(\alpha P)$

$\ge \int(\mathrm{Cr}\,x\alpha \bullet .\,fx)\varphi\,\mathrm{d}\,x + \int(\lambda \bullet \mathrm{Cr}\,x\alpha \bullet \mathrm{Cr}\,xP)\varphi\,\mathrm{d}\,x)$

$= \int(\mathrm{Cr}\,x\alpha \bullet (.\,fx + \lambda \bullet \mathrm{Cr}\,xP)\varphi\,\mathrm{d}\,x$

$= \int(\mathrm{Cr}\,x\alpha \bullet .\,gx)\varphi\,\mathrm{d}\,x$

$= .\,\mathrm{Gr}\,g\varphi\alpha)$.

Hence

$(g \in H)$,

and because of .5, .3, .15, and .0

$(N$

$\le N + \lambda \bullet .\,\varphi P$

$= \int(\mathrm{Cr}\,xA \bullet .\,fx)\varphi\,\mathrm{d}\,x + \int(\lambda \bullet \mathrm{Cr}\,xA \bullet \mathrm{Cr}\,xP)\varphi\,\mathrm{d}\,x)$

$= \int(\mathrm{Cr}\,xA \bullet (.\,fx + \lambda \bullet \mathrm{Cr}\,xP)\varphi\,\mathrm{d}\,x$

$= \int(\mathrm{Cr}\,xA \bullet .\,gx)\varphi\,\mathrm{d}\,x$

$\le N)$

According to this and .1

$(\lambda \cdot .\,\varphi P = 0)$

and hence

.17 $(\lambda = 0 \vee .\,\varphi P = 0)$.

Because of .17, .9, and .14

$(.\,\psi A = 0)$.

Therefore, because of .11 and .8

$(\alpha \in M \to 0 = .\,\psi\alpha = .\,mu\alpha — .\,\mu'\alpha \to .\,\mu\alpha = .\,\mu'\alpha)$

and because of .4 and .3 the desired conclusion is at hand.

Lemma

4.29 $(A \in \mathrm{mbl}\,\mu\,\mathrm{mbl}'\,\varphi \wedge M = \mathrm{mbl}\,\varphi\,\mathrm{sb}\,A \wedge \mathrm{mbl}\,\varphi \subset \mathrm{mbl}\,\mu \wedge \mathrm{zr}\,\varphi \subset \mathrm{zr}\,\mu$

$\to \bigvee f \in \mathrm{ff}{+}\,\mathrm{mbl}\,\varphi(\mathrm{strc}\,\mu M = \mathrm{strc}\,\mathrm{Gr}\,f\varphi M))$

Proof:

Let

$(N = \sup \alpha \in M \cap \mathrm{dmn}'\,\mu\,.\,\varphi\alpha)$.

Clearly

$(0 \le N \le .\,\varphi A < \infty)$.

Next choose

$(T \in \mathrm{sqnc}{\subset}(M \cap \mathrm{dmn}'\,\mu))$

so that

$(.\,T0 = 0 \wedge \mathrm{lin}\,n\,.\,\varphi\,.\,Tn = N)$

and let

$(B = \bigvee n \in \omega\,.\,Tn \wedge C = A{\sim}B \wedge S = \curlywedge n \in \omega(.\,T(n+1){\sim}.\,Tn))$.

Clearly

$(B = \bigvee n \in \omega \,.\, Sn)$

(in a disjointed way). We take advantage of 4.28 to secure such a

$(\xi \in \mathrm{sqnc}\,\mathrm{ff}{+}\,\mathrm{mbl}\,\varphi)$

that

$\bigwedge n \in \omega \bigwedge \alpha \in M(\,.\,\mu(\alpha \,.\, Sn) = \int(\mathrm{Cr}\,x(\alpha \,.\, Sn) \bullet .\,.\,\xi nx)\varphi \,\mathrm{d}\,x)$

and let

$(f = \curlywedge x \in \mathrm{rlm}\,\varphi(\mathrm{Cr}\,xC \bullet \infty + \sum n \in \omega(\mathrm{Cr}\,x \,.\, Sn \bullet .\,.\,\xi nx)))$.

Obviously

$(f \in \mathrm{ff}{+}\,\mathrm{mbl}\,\varphi)$.

Now

$(\alpha \in M \wedge .\,\mu(\alpha C) < \infty \wedge n \in \omega \rightarrow .\,\varphi \,.\, Tn \leq .\,\varphi \,.\, Tn + .\,\varphi(\alpha C) = .\,\varphi(.\,Tn \cup \alpha C) \leq N)$

and hence

$(\alpha \in M \wedge .\,\mu(\alpha C) < \infty \rightarrow N \leq N + .\,\varphi(\alpha C) \leq N \wedge .\,\varphi(\alpha C) = 0 \wedge .\,\mu(\alpha C) = 0)$.

Consequently it is certain that

$(\alpha \in M \rightarrow$

$.\,\mu(\alpha C) = \int(\mathrm{Cr}\,x(\alpha C) \bullet \infty)\varphi \,\mathrm{d}\,x \wedge$

$.\,\mu\alpha$

$= .\,\mu(\alpha C) + .\,\mu(\alpha B)$

$= .\,\mu(\alpha C) + \sum n \in \omega \,.\, \mu(\alpha \,.\, Sn)$

$= \int(\mathrm{Cr}\,x(\alpha C) \bullet \infty)\varphi \,\mathrm{d}\,x + \sum n \in \omega \int(\mathrm{Cr}\,x(\alpha \,.\, Sn) \bullet .\,.\,\xi nx)\varphi \,\mathrm{d}\,x$

$= \int(\mathrm{Cr}\,x\alpha \bullet (\mathrm{Cr}\,xC \bullet \infty + \sum n \in \omega(\mathrm{Cr}\,x \,.\, Sn \bullet .\,.\,\xi nx)))\varphi \,\mathrm{d}\,x$

$= \int(\mathrm{Cr}\,x\alpha \,.\, fx)\varphi \,\mathrm{d}\,x$

$= .\,\mathrm{Gr}\,f\varphi\alpha)$

Theorem

4.30 $(A \in \mathrm{mbl}\,\mu\,\mathrm{mbl}''\,\varphi \wedge M = \mathrm{mbl}\,\varphi\,\mathrm{sb}\,A \wedge \mathrm{mbl}\,\varphi \subset \mathrm{mbl}\,\mu \wedge \mathrm{zr}\,\varphi \subset \mathrm{zr}\,\mu$
$\rightarrow \bigvee f \in \mathrm{ff}{+}\,\mathrm{mbl}\,\varphi(\mathrm{strc}\,\mu M = \mathrm{strc}\,\mathrm{Gr}\,f\varphi M))$

Radon-Nikodym Theorem

4.31 $(\mu \in \mathrm{Msr}\,S \wedge \varphi \in \mathrm{Msr}\,S \wedge S \in \mathrm{mbl}''\,\varphi \wedge \mathrm{mbl}\,\varphi \subset \mathrm{mbl}\,\mu \wedge \mathrm{zr}\,\varphi \subset \mathrm{zr}\,\mu \rightarrow \bigvee f \in \mathrm{ff}{+}\,\mathrm{mbl}\,\varphi(\mathrm{Gr}\,f\varphi \subset \mu))$

Theorem

4.32 $(\mathrm{massile}\,\varphi x\underline{\mathrm{u}}x \wedge \mathrm{massile}\,\varphi x\underline{\mathrm{v}}x \wedge \bigwedge \alpha \in \mathrm{mbl}'\,\varphi(\int \alpha; \underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x \leq \int \alpha; \underline{\mathrm{v}}x\varphi \,\mathrm{d}\,x) \rightarrow \mathrm{Alm}\,\varphi x(\underline{\mathrm{u}}x \leq \underline{\mathrm{v}}x))$

Proof:

Let

$(S = \mathrm{rlm}\,\varphi)$

and divide the rest of the proof into 4 parts.

Part 0:

$(A \in \mathrm{mbl}'\,\varphi \wedge r, s, \in \mathrm{rn} \wedge B = \mathsf{E}\, x \in A(\underline{\mathrm{v}}x < r < s < \underline{\mathrm{u}}x) \rightarrow B \in \mathrm{zr}\,\varphi)$

Proof:

We may as well assume

$(-\infty < r < s < \infty)$.

Now

$(-\infty \leq \int B; \underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x \leq \int B; \underline{\mathrm{v}}x\varphi \,\mathrm{d}\,x \leq r \cdot .\,\varphi B \leq s \cdot .\,\varphi B \leq \int B; \underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x \leq \infty \wedge$

$s \cdot .\,\varphi B - r \cdot .\,\varphi B = 0 \wedge$

$(s - r) \cdot .\,\varphi B = 0 \wedge$

$.\,\varphi B = 0)$

Part 1:

$(A \in \mathrm{mbl}'\,\varphi \rightarrow \mathsf{E}\, x \in A(\underline{\mathrm{v}}x \le \underline{\mathrm{u}}x) \in \mathrm{zr}\,\varphi)$

Proof:

$(\mathsf{E}\, x \in A(\underline{\mathrm{v}}x \le \underline{\mathrm{u}}x) = \bigvee r, s, \in \mathrm{rn}\, \mathsf{E}\, x \in A(\underline{\mathrm{v}}x < s < r < \underline{\mathrm{u}}x) \in \mathrm{zr}\,\varphi)$

Part 2:

$(A \in \mathrm{mbl}'\,\varphi \rightarrow A{\sim}\,\mathsf{E}\, x(\underline{\mathrm{u}}x \le \underline{\mathrm{v}}x) \in \mathrm{zr}\,\varphi)$

Proof:

$(A{\sim}\,\mathsf{E}\, x(\underline{\mathrm{u}}x \le \underline{\mathrm{v}}x) = A{\sim}\,\mathsf{E}\, x(\underline{\mathrm{u}}xin\,\mathrm{rl}) \cup A{\sim}\,\mathsf{E}\, x(\underline{\mathrm{v}}x \in \mathrm{rl}) \cup \mathsf{E}\, x \in A(\underline{\mathrm{v}}x < \underline{\mathrm{u}}x) \in \mathrm{zr}\,\varphi)$

Part 3:

$\mathrm{Alm}\,\varphi x(\underline{\mathrm{u}}x \le \underline{\mathrm{v}}x)$

Proof:

Secure, as can clearly be done, such a

$(F \in \mathrm{cbl} \cap \mathrm{sb}\,\mathrm{mbl}'\,\varphi)$

that

$(\nabla F = \mathsf{E}\, x \in S(|\underline{\mathrm{u}}x| > 0) \cup \mathsf{E}\, x \in S(|\underline{\mathrm{v}}x| > 0))$

and note that

$(S{\sim}\,\mathsf{E}\, x(\underline{\mathrm{u}}x \le \underline{\mathrm{v}}x) = \bigvee \beta \in F(\beta{\sim}\,\mathsf{E}\, x(\underline{\mathrm{u}}x \le \underline{\mathrm{v}}x)))$.

Theorem

4.33 $(\mathrm{massile}\,\varphi x \underline{\mathrm{u}}x \wedge \mathrm{massile}\,\varphi x \underline{\mathrm{v}}x \wedge \bigwedge \alpha \in \mathrm{mbl}'\,\varphi(\int \alpha; \underline{\mathrm{u}}x\varphi\,\mathrm{d}\,x = \int \alpha; \underline{\mathrm{v}}x\varphi\,\mathrm{d}\,x) \rightarrow \mathrm{Alm}\,\varphi x(\underline{\mathrm{u}}x = \underline{\mathrm{v}}x))$

Theorem

4.34 $(\theta \in \mathrm{addor\,gauge} \wedge \mathrm{dmn}\,\theta \in \mathrm{bsc}\,\varphi \wedge S = \mathrm{rlm}\,\theta = \mathrm{rlm}\,\varphi \wedge \mathrm{zr}\,\varphi \cap \mathrm{dmn}\,\theta \subset \mathrm{zr}\,\theta \wedge \psi = \mathrm{mr}\,\theta$
$\rightarrow \theta \subset \psi \in \mathrm{Msr}\,S \wedge \mathrm{mbl}\,\varphi \subset \mathrm{mbl}\,\psi \wedge \mathrm{zr}\,\varphi \subset \mathrm{zr}\,\psi)$

Proof:

Let

$(M = \mathrm{dmn}\,\theta)$,

note that

$\mathrm{grated\,is}\,M$,

use 4.4 and 2.16 in checking

$(\theta \subset \psi \in \mathrm{Msr}\,S \wedge M \subset \mathrm{mbl}\,\psi)$,

and divide the rest of the proof into six parts.

Part 0:

$(A \in M \wedge B \in \mathrm{zr}\,\varphi \rightarrow BA \in \mathrm{zr}\,\psi)$

Proof:

Choose

$(A' \in M \cap \mathrm{sb}\,A \cap \mathrm{sp}(BA))$

so that

$(\centerdot\,\varphi(A') = 0)$.

We then have

$(\centerdot\,\psi(BA) \le \centerdot\,\psi A' \le \centerdot\,\theta A' = 0)$.

Part 1:

$(A \in \mathrm{zr}\,\varphi \rightarrow A \in \mathrm{zr}\,\psi)$

Proof:

Choose

$(F \in \mathrm{cuv}\,AM)$

so that

$(\sum \beta \in F\,\centerdot\,\varphi\beta \le 1)$.

Because of Part 0,

$(A = \bigvee \beta \in F(\beta A) \in \mathrm{zr}\,\psi)$.

Part 2:

$(A \in M \cap \mathrm{mbl}'\,\varphi \wedge B \in \mathrm{mbl}\,\varphi \rightarrow BA \in \mathrm{mbl}\,\psi)$

Proof:

Choose

$(A' \in M \cap \mathrm{sb}\,A \cap \mathrm{sp}(BA))$

so that

$(\boldsymbol{.}\,\varphi(BA) = \boldsymbol{.}\,\varphi A')$.

Clearly

$(\boldsymbol{.}\,\varphi(A' \sim (BA)) = \boldsymbol{.}\,\varphi A' - \boldsymbol{.}\,\varphi(BA) = 0 \wedge$
$A' \sim (BA) \in \mathrm{zr}\,\psi \cap \mathrm{mbl}\,\psi \wedge$
$BA = A' \sim (A' \sim (BA)) \in \mathrm{mbl}\,\psi)$

Part 3:

$(A \in M \wedge B \in \mathrm{mbl}\,\varphi \rightarrow BA \in \mathrm{mbl}\,\psi)$

Proof:

Secure, as can clearly be done, such an

$(F \in \mathrm{cbl} \cap \mathrm{sb}(M\,\mathrm{mbl}'\,\varphi))$

that

$(A \subset \nabla F)$.

According to Part 2,

$(BA = \bigvee \alpha \in F(B\alpha) \in \mathrm{mbl}\,\psi)$.

Part 4:

$(B \in \mathrm{mbl}\,\varphi \rightarrow B \in \mathrm{mbl}\,\psi)$

Proof:

Because of Part 3

$(A \in m \rightarrow AB \in \mathrm{mbl}\,\psi \rightarrow \boldsymbol{.}\,\psi A = \boldsymbol{.}\,\psi(AB) + \boldsymbol{.}\,\psi(A \sim B))$.

Now use 2.6.2.

Part 5:

$(\mathrm{zr}\,\varphi \subset \mathrm{zr}\,\psi \wedge \mathrm{mbl}\,\varphi \subset \mathrm{mbl}\,\psi)$

Proof:

Use Part 1 and Part 4.

With the help of 4.34 and 4.31 we now have

Theorem

4.35 $(\theta \in \mathrm{addor\,gauge} \wedge \varphi \in \mathrm{Hilt\,dmn}\,\theta \wedge \mathrm{zr}\,\varphi \cap \mathrm{dmn}\,\theta \subset \mathrm{zr}\,\theta \rightarrow \bigvee f \in \mathrm{ff}{+}\,\mathrm{mbl}\,\varphi(\theta \subset \mathrm{Gr}\,f\varphi))$

Helped by the Jordan decomposition theorem we easily infer

Theorem

4.36 $(\psi \in \mathrm{addor} \wedge \varphi \in \mathrm{Hilt\,dmn}\,\psi \wedge \mathrm{zr}\,\varphi \cap \mathrm{dmn}\,\psi \subset \mathrm{zr}\,\psi \rightarrow \bigvee f \in \mathrm{ff\,mbl}\,\varphi(\psi \subset \mathrm{Gr}\,f\varphi))$

Still more general than 4.35 and not well known is

Theorem

4.37 $(\theta \in \mathrm{addor\,gauge} \wedge \varphi \in \mathrm{Msr}\,S \wedge \mathrm{dmn}\,\theta \subset \mathrm{mbl}\,\varphi \wedge \mathrm{zr}\,\varphi \cap \mathrm{dmn}\,\theta \subset \mathrm{zr}\,\theta \wedge S \in \nabla''(\mathrm{dmn}\,\theta \cap \mathrm{dmn}'\,\varphi)$
$\rightarrow \bigvee f \in \mathrm{ff}{+}\,\mathrm{mbl}\,\varphi(\theta \subset \mathrm{Gr}\,f\varphi))$

Let

$(M = \mathrm{dmn}\,\theta \wedge \psi = \mathrm{mr\,strc}\,\varphi M)$.

Note that

$\qquad$ grated is M ,

use 4.4 and 2.26 in checking

$\qquad (\bigwedge \beta \in M(\mathbf{.}\,\psi\beta = \mathbf{.}\,\varphi\beta) \wedge M \subset \operatorname{mbl}\psi)$,

and then observe that

$\qquad (S \in \operatorname{mbl}''\psi \wedge \psi \in \operatorname{Hilt} M \wedge \operatorname{zr}\psi \cap M \subset \operatorname{zr}\theta)$.

Now, in accordance with 4.35, we so choose

$\qquad (f \in \operatorname{ff+} \operatorname{mbl}\psi)$

that

$\qquad (\theta \subset \operatorname{Gr} f\psi)$.

Because of 2.65

$\qquad (\operatorname{mbl}\psi \subset \operatorname{mbl}\varphi \wedge \operatorname{mbl}'\psi \subset \operatorname{mbl}'\varphi \wedge \bigwedge \beta \in \operatorname{mbl}'\psi(\mathbf{.}\,\psi\beta = \mathbf{.}\,\varphi\beta))$.

Hence

$\qquad (f \in \operatorname{ff+} \operatorname{mbl}\varphi)$

and, because of 3.96,

$\qquad \bigwedge \beta \in M(\operatorname{rl} \supset \mathbf{.}\,\theta\beta = \mathbf{.}\operatorname{Gr} f\psi\beta = \int(\operatorname{Cr} x\beta \bullet \mathbf{.}\, fx)\psi \,\mathrm{d}x = \int(\operatorname{Cr} x\beta \bullet \mathbf{.}\, fx)\varphi \,\mathrm{d}x = \mathbf{.}\operatorname{Gr} f\varphi\beta)$.

Thus

$\qquad (f \in \operatorname{ff+} \operatorname{mbl}\varphi \wedge \theta \subset \operatorname{Gr} f\varphi)$

and the proof is finished.

Helped again by the Jordan decomposition theorem we easily infer

Theorem

4.38 $(\psi \in \operatorname{addor} \wedge \varphi \in \operatorname{Msr} S \wedge \operatorname{dmn}\psi \subset \operatorname{mbl}\varphi \wedge \operatorname{zr}\varphi \cap \operatorname{dmn}\psi \subset \operatorname{zr}\psi \wedge S \in \nabla''(\operatorname{dmn}\psi \cap \operatorname{dmn}'\varphi)$

$\qquad \rightarrow \bigvee f \in \operatorname{ff} \operatorname{mbl}\varphi(\psi \subset \operatorname{Gr} f\varphi))$

In neither 4.37 nor 4.38 is the f necessarily unique.

Theorem

4.39 $(f \in \operatorname{ff} \operatorname{mbl}\varphi \wedge \varphi \in \operatorname{Hilt} \operatorname{dmn}\psi \wedge f_1 = \measuredangle x \in \operatorname{rlm}\varphi \operatorname{ps}\mathbf{.}\, fx \wedge \psi \subset \operatorname{Gr} f\varphi \wedge \theta_1 = \operatorname{dor+}\psi \wedge \psi_1 = \operatorname{mr}\theta_1$

$\qquad \rightarrow f_1 \in \operatorname{ff+} \operatorname{mbl}\varphi \wedge \operatorname{Gr} f_1\varphi \subset \psi_1)$

Theorem

4.40 $(f \in \operatorname{ff} \operatorname{mbl}\varphi \wedge g \in \operatorname{ff} \operatorname{mbl}\varphi \wedge \varphi \in \operatorname{Hilt} \operatorname{dmn}\psi \wedge \psi \subset \operatorname{Gr} f\varphi \cap \operatorname{Gr} g\psi \rightarrow \operatorname{Alm}\varphi x(\mathbf{.}\, fx = \mathbf{.}\, gx))$

Measure and Integration: Word Index

This index is to the symbols made up primarily as letter combinations. They are listed alphabetically with a reference to their first location in the Measure and Integration notes. “BN” references are to Background Notation.

Measure and Integration: Symbol Index

This index is to the non-word symbols. They are listed in the order of their first appearance in the Measure and Integration notes. "BN" references are to Background Notation.

Symbol	Location	Meaning
(	BN.0	left parenthesis
∧	BN.0	and
↔	BN.0	if and only if
)	BN.0	right parenthesis
∨	BN.1	or
∼	BN.2	not / complement
⋀	BN.3	for each / indexed intersection
,	BN.3	comma / ordered pair
⋁	BN.4	for some / indexed union
∈	BN.5	belongs to, is a member of
→	BN.5	implies
∋	BN.7	holds, has as a member
∽	BN.8	set difference, set minus
⊂	BN.9	is a subset of
⊂·	BN.10	is a proper subset of
≠	BN.10	not equal
∩	BN.11	intersect (binary)
=	BN.11	equals, is equal to
E	BN.15	classifier, set of
{	BN.15	left brace
:	BN.15	colon
}	BN.15	right brace
∇	BN.16	union of
⊓	BN.17	intersection of
0	BN.19	false / empty set, zero
$_*$	BN.34	image; $_*RA$
*	BN.35	inverse image; $^*RB$
.	BN.47	function value; **.** fx
⅄	BN.48	function builder ("lonzo")
ω	BN.51	set of natural numbers beginning with 0
1	BN.52	one
2	BN.52	two
3	BN.52	three
+	BN.52	plus (addition)
∪	BN.52	union (binary)
$\vert$	BN.56	absolute value bar; $\vert x\vert$
∞	BN.56	infinity
−	BN.57	negative
$\leq$	BN.57	less than or equal to
$<$	BN.58	less than
$\sum$	BN.60	sum
$\surd$	BN.61	square root
$\cdot$	BN.61	times
/	BN.65	divided by
−	BN.71	minus (subtraction)
$\underline{2}$	BN.80	1/2 to an integral power

$^{\bullet}$	BN.50	raise to an integral power; $^{\bullet}xn$
$\geq$	BN.87	greater than or equal to
$>$	R.0.8	greater than
ω'	R.8.0	set of integers
$\bullet$	R.21	heavy times, same as $x \cdot y$, except $(0 \bullet x = 0)$ for *any* x
$\sum$	R.48	symmetric sum
$\sim'$	2.0.3	set of complements
∇''	2.0.4	countable unions
$\sqcap''$	2.0.5	countable intersections
∇field	2.0.6	sigma field
$\subset\subset$	2.50	is a refinement of
$\underline{\cap}$	3.43.1	common refinement
$\underline{⨍}$	3.48.0	central integral
$\overline{\int}$	3.48.1	upper integral
$\underline{\int}$	3.48.2	lower integral
$\overline{\int}\#$	3.49	preliminary integral
d	3.49	integration d
$\int$	3.68	integral

Seminar Integration (1963)

Table of Contents

1. Some Elements of Analysis

In the definitions and postulates which follow a fundamental role is played by the constants
‘+’, ’·’, ‘−’, ‘/’, ‘Sup’, ‘i’ .

1.0 Definitions

.0 $(1 \equiv \mathrm{scsr}\, 0)$
.1 $(2 \equiv \mathrm{scsr}\, 1)$
.2 $(3 \equiv \mathrm{scsr}\, 2)$
.3 $(4 \equiv \mathrm{scsr}\, 3)$
.4 $(5 \equiv \mathrm{scsr}\, 4)$
.5 $(6 \equiv \mathrm{scsr}\, 5)$
.6 $(7 \equiv \mathrm{scsr}\, 6)$
.7 $(8 \equiv \mathrm{scsr}\, 7)$
.8 $(9 \equiv \mathrm{scsr}\, 8)$

1.1 Definitions

.0 $(\omega \equiv \sqcap \sqsubseteq A(0 \in A \wedge \bigwedge n \in A(\mathrm{scsr}\, n \in A)))$
.1 $(\text{The set of natural numbers} \equiv \omega)$
.2 $((x - y) \equiv (x + - y))$
.3 $(\mathrm{kf} \equiv \sqsubseteq x(x - x = 0))$
.4 $(\mathrm{infin} \equiv \sqsubseteq x(1/x = 0))$
.5 $(\mathrm{kt} \equiv (\mathrm{kf} \cup \mathrm{infin}))$
.6 $(\mathrm{complextension} \equiv \mathrm{kt})$
.7 $(⦶ \equiv (1/0))$
.8 $(\mathrm{cp} \equiv (\mathrm{kf} \cup \mathrm{sng}\, ⦶))$
.9 $(\mathrm{complexplane} \equiv \mathrm{cp})$
.10 $(\mathrm{rl} \equiv \sqsubseteq x \in \mathrm{kt}(\mathrm{Sup}\,\mathrm{sng}\, x \neq \mathrm{U}))$
.11 $(\mathrm{rf} \equiv (\mathrm{kf} \cap \mathrm{rl}))$
.12 $(\mathrm{rp} \equiv \sqsubseteq x \in \mathrm{rl} \backsim 1(\mathrm{Sup}(\mathrm{sng}\, 0 \cup \mathrm{sng}\, x) \neq 0))$
.13 $(\mathrm{rfp} \equiv (\mathrm{rf} \cap \mathrm{rp}))$
.14 $((x < y) \equiv (x \in \mathrm{rl} \wedge y \in \mathrm{rl} \wedge y - x \in \mathrm{rp}))$
.15 $((x \leq y) \equiv (x < y \vee x = y \in \mathrm{rl}))$
.16 $((x > y) \equiv (y < x))$
.17 $((x \geq y) \equiv (y \leq x))$
.18 $(\infty \equiv \mathrm{Sup}\,\mathrm{rl}))$
.19 $(\mathrm{dinfin} \equiv \bigvee z \in \mathrm{kf} \backsim 1\, \mathrm{sng}(z \cdot \infty))$
.20 $(\mathrm{directedinfinities} \equiv \mathrm{dinfin})$
.21 $(\mathrm{spl} \equiv (\mathrm{kf} \cup \mathrm{rl}))$
.22 $(\mathrm{summationplane} \equiv \mathrm{spl})$

1.2 Postulates

.0 $(0 \in \omega)$
.1 $(x \in \omega \to \mathrm{scsr}\, x = x + 1 \in \omega)$
.2 $(0 \in S \subset \omega \wedge \bigwedge n \in S(n + 1 \in S) \to S = \omega)$

1.3 Postulates

.0 $(\mathrm{kt} \in \mathrm{U})$

.1 $(\mathrm{infin} = \mathrm{dinfin} \cup \mathrm{sng}\, \infty\!\!\!\!\!\mid\,)$

.2 $(\mathrm{rl} = \mathrm{rf} \cup \mathrm{sng}\, \infty \cup \mathrm{sng} - \infty)$

.3 $(x \in \mathrm{kf} \wedge y \in \mathrm{infin} \rightarrow x + y = y)$

.4 $(x \in \mathrm{kf} \wedge y \in \mathrm{kf} \wedge z \in \mathrm{infin} \wedge x/y \backsim\in \mathrm{rfp} \rightarrow x \cdot z + y \cdot z = \mathrm{U})$

.5 $(0 \neq x \in \mathrm{kt} \rightarrow x \cdot \infty\!\!\!\!\!\mid\, = \infty\!\!\!\!\!\mid\,)$

1.4 Postulates

.0 $((x + y \neq \mathrm{U} \rightarrow x \in \mathrm{U} \wedge x + y \in \mathrm{U}) \wedge$
$(x \cdot y \neq \mathrm{U} \rightarrow x \in \mathrm{U} \wedge x \cdot y \in \mathrm{U}) \wedge$
$(1/x \neq \mathrm{U} \rightarrow x \in \mathrm{U} \wedge 1/x \in \mathrm{U}))$

.1 $(x + y \neq \mathrm{U} \rightarrow x \in \mathrm{kt} \wedge y \in \mathrm{kt} \leftrightarrow x + y \in \mathrm{kt})$

.2 $(x \cdot y \neq \mathrm{U} \rightarrow x \in \mathrm{kt} \wedge y \in \mathrm{kt} \leftrightarrow x \cdot y \in \mathrm{kt})$

.3 $(x \in \mathrm{kt} \leftrightarrow -x \in \mathrm{kt})$

.4 $(x \in \mathrm{kt} \leftrightarrow 1/x \in \mathrm{kt})$

.5 $(\mathrm{Sup}\, A \neq \mathrm{U} \rightarrow A \subset \mathrm{rl})$

.6 $(A \subset \mathrm{rl} \rightarrow \mathrm{Sup}\, A \in \mathrm{rl})$

1.5 Postulates

.0 $(\omega \subset \mathrm{rf})$

.1 $(z \in \mathrm{kf} \leftrightarrow \bigvee x \in \mathrm{rf} \bigvee y \in \mathrm{rf}(z = x + \mathrm{i} \cdot y))$

.2 $(x \in \mathrm{rf} \wedge y \in \mathrm{rf} \rightarrow x + \mathrm{i} \cdot y = 0 \leftrightarrow x = 0 \wedge y = 0)$

.3 $(\mathrm{i} \cdot \mathrm{i} = -1)$

1.6 Postulates

.0 $(x \in \mathrm{kt} \rightarrow 1 \cdot x = x = 0 + x)$

.1 $(-x = -1 \cdot x)$

.2 $(x/y = x \cdot (1/y))$

.3 $(x \in \mathrm{rf} \wedge y \in \mathrm{rf} \rightarrow x + y \in \mathrm{rf} \wedge x \cdot y \in \mathrm{rf})$

.4 $(x \in \mathrm{rp} \wedge y \in \mathrm{rp} \rightarrow x + y \in \mathrm{rp} \wedge x \cdot y \in \mathrm{rp})$

.5 $(0 \neq x \in \mathrm{rl} \rightarrow x \in \mathrm{rp} \vee -x \in \mathrm{rp})$

.6 $(x + y = y + x \wedge x \cdot y = y \cdot x)$

.7 $((x + y) + z = x + (y + z))$

.8 $((x \cdot y) \cdot z = x \cdot (y \cdot z))$

.9 $(z \in \mathrm{kf} \rightarrow z \cdot (x + y) = z \cdot x + z \cdot y)$

.10 $(0 \neq x \in \mathrm{kf} \rightarrow x/x = 1)$

.11 $(A \subset \mathrm{rl} \wedge t \in \mathrm{rl} \rightarrow \bigwedge x \in A(y \leq t) \leftrightarrow \mathrm{Sup}\, A \leq t)$

1.7 Definitions

.0 $(\omega' \equiv \mathsf{E}\, n(n \in \omega \vee -n \in \omega))$

.1 (The set of integers $\equiv \omega'$)

1.8 Postulates

.0 $(x \in \mathrm{kt} \rightarrow {}^{\bullet}x0 = 1)$

.1 $(x \in \mathrm{kt} \wedge n \in \omega \rightarrow {}^{\bullet}x(n + 1) = x \cdot {}^{\bullet}xn)$

.2 $(n \in \omega' \sim \omega \rightarrow {}^{\bullet}xn = {}^{\bullet}(1/x) - n)$

.3 $({}^{\bullet}xn \neq \mathrm{U} \rightarrow n \in \omega' \wedge x \in \mathrm{U})$

Definition
1.9 $(\mathrm{noz}\, x \equiv (x = 0 \vee x))$

1.10 Postulates

.0 $(x \in \mathrm{kt} \wedge 0 \neq y \in \mathrm{To}\,\mathrm{kt} \rightarrow x + y = \mathrm{noz} \curlywedge t(x + .yt) \wedge x \cdot y = \mathrm{noz} \curlywedge t(x \cdot .yt))$

.1 $(0 \neq x \in \mathrm{To}\,\mathrm{kt} \wedge 0 \neq y \in \mathrm{To}\,\mathrm{kt} \rightarrow x + y = \mathrm{noz} \curlywedge t(.xt + .yt) \wedge x \cdot y = \mathrm{noz} \curlywedge t(.xt \cdot .yt))$

1.11 Theorems

.0 $(\omega \subset \mathrm{rf} \subset \mathrm{kf} \subset \mathrm{kt})$

.1 $(\mathrm{rfp} \subset \mathrm{rf} \subset \mathrm{rl} \subset \mathrm{kt})$

.2 $(\mathrm{rfp} \subset \mathrm{rp} \subset \mathrm{rl})$

.3 $(\infty \in \mathrm{rl} \wedge -\infty \in \mathrm{rl})$

.4 $(x \in \mathrm{kf} \rightarrow x — x = 0)$

.5 $(0 \in \mathrm{rf} \wedge 1 \in \mathrm{rf})$

.6 $(x \in \mathrm{kf} \rightarrow 0 \cdot x = 0)$

Proof:

$(0 = x — x = x \cdot 1 + x \cdot -1 = x \cdot (1 — 1) = x \cdot 0 = 0 \cdot x)$

.7 $(x \in \mathrm{kf} \rightarrow -x \in \mathrm{kf})$

Proof:

Because of 1.3.3

$(x \in \mathrm{kf} \wedge -x \in \mathrm{infin} \rightarrow 0 = x — x = -x \wedge -x \in \mathrm{kf})$

.8 $(\mathrm{i} \in \mathrm{kt})$

Proof:

$(\mathrm{i} \cdot \mathrm{i} = -1 \in \mathrm{kt} \wedge \mathrm{i} \in \mathrm{kt})$

.9 $(\mathrm{i} \in \mathrm{kf})$

Proof;

$(0 \in \mathrm{rf} \wedge 1 \in \mathrm{rf} \wedge \mathrm{i} = 0 + \mathrm{i} \cdot 1)$

.10 $(x \in \mathrm{rf} \rightarrow -x \in \mathrm{rf})$

Proof:

Since

$(-x \in \mathrm{kf})$

we can and do choose such members a and b of rf that

$(-x = a + \mathrm{i} \cdot b)$.

Thus

$(0 = x — x = (x + a) + \mathrm{i} \cdot b \wedge b = 0 \wedge -x = a \in \mathrm{rf})$

.11 $(-1 \cdot -1 = 1)$

.12 $(x \in \mathrm{kt} \rightarrow - - x = x)$

.13 $(x \in \mathrm{rl} \rightarrow -x \in \mathrm{rl})$

.14 $((x \in \mathrm{kt} \leftrightarrow -x \in \mathrm{kt}) \wedge (x \in \mathrm{kf} \leftrightarrow -x \in \mathrm{kf}) \wedge (x \in \mathrm{rl} \leftrightarrow -x \in \mathrm{rl}) \wedge (x \in \mathrm{rf} \leftrightarrow -x \in \mathrm{rf}))$

.15 $\sim(0 \in \mathrm{rp})$

.16 $(1 \in \mathrm{rp})$

.17 $\sim(-1 \in \mathrm{rp})$

.18 $\sim(-1 = 0)$

.19 $(1/-1 = -1)$

.20 $(z \in \mathrm{infin} \rightarrow z — z = \mathrm{U})$

.21 $(\mathrm{infin} \cap \mathrm{kf} = 0)$

.22 $(x \in \mathrm{infin} \rightarrow x + y \backsim\in \mathrm{kf})$

.23 $(\infty \in \mathrm{dinfin} \subset \mathrm{infin} \wedge -\infty \in \mathrm{dinfin})$

.24 $(\infty + \infty \in \mathrm{rl})$

.25 $(\infty + \infty \neq ⧞)$
.26 $(\infty + \infty = \infty)$
.27 $(0 \cdot \infty = \mathrm{U})$
.28 $(0 \cdot ⧞ = \mathrm{U})$
Proof:
$(0 \cdot ⧞ = 0 \cdot (\infty \cdot ⧞) = (0 \cdot \infty) \cdot ⧞ = \mathrm{U})$
.29 $(\mathrm{infin} = \mathsf{E}\, x \in \mathrm{kt}(0 \cdot x = \mathrm{U}))$
.30 $(\mathrm{kf} = \mathsf{E}\, x(0 \cdot x = 0) = \mathsf{E}\, x(0 \cdot x \in \mathrm{kt}) = \mathsf{E}\, x(x - x = 0) = \mathsf{E}\, x(x - x \in \mathrm{kt}))$
.31 $(x + y \in \mathrm{kf} \leftrightarrow x \in \mathrm{kf} \wedge y \in \mathrm{kf} \leftrightarrow x \cdot y \in \mathrm{kf})$
.32 $(z \in \mathrm{dinfin} \to z + z = z)$
.33 $(⧞ + ⧞ = \mathrm{U})$
Proof:
$(⧞ + ⧞ = ⧞ + (-1 \cdot ⧞) = ⧞ - ⧞ = \mathrm{U})$
.34 $\sim(⧞ \in \mathrm{dinfin})$
.35 $(\sim(\infty + ⧞ \in \mathrm{dinfin})$
.36 $(\infty + ⧞ = \mathrm{U})$
.37 $(z \in \mathrm{dinfin} \to z + ⧞ = \mathrm{U})$
.38 $(x \cdot y = 0 \to x \in \mathrm{kf})$
Proof:
$(x \cdot y = 0 \to (0 \cdot x) \cdot y = 0 \to 0 \cdot x \in \mathrm{kt} \to x \in \mathrm{kf})$
.39 $(0 \neq x \in \mathrm{kf} \wedge a = 1/x \to 0 \neq a \in \mathrm{kf} \wedge 1/a = x)$
Proof:
$(x \cdot a = 1 \wedge a \neq 0 \wedge 0 \cdot (x \cdot a) = 0 \wedge 0 \neq a \in \mathrm{kf} \wedge x = (x \cdot a) \cdot (1/a) = 1 \cdot (1/a) = 1/a)$
.40 $(x \cdot y = 0 \to x = 0 \vee y = 0)$
Proof:
$(x \in \mathrm{kf} \wedge x \neq 0 \to y = (1/x) \cdot (x \cdot y) = (1/x) \cdot 0 = 0)$
.41 $((0 \neq x \in \mathrm{rf} \to 1/x \in rf) \wedge (x \in \mathrm{rp} \to 1/x \in \mathrm{rp}) \wedge (0 \neq x \in \mathrm{rl} \to 1/x \in \mathrm{rl}))$
.42 $\sim(x \in \mathrm{rp} \wedge -x \in \mathrm{rp})$
.43 $(0 < x \leftrightarrow x \in \mathrm{rp})$
.44 $(a < b \leftrightarrow -b < -a)$
.45 $(a \leq b \leftrightarrow -b \leq -a)$
.46 $(a < b < c \to a < c)$
.47 $(a < b \leq c \to a < c)$
.48 $(a \leq b < c \to a < c)$
.49 $(a \leq b \leq c \to a \leq c)$
.50 $(\mathrm{rl} = \mathsf{E}\, x(x \leq \infty))$
.51 $(\mathrm{rl} = \mathsf{E}\, x(-\infty \leq x \leq \infty))$
.52 $(\mathrm{rf} = \mathsf{E}\, x(-\infty < x < \infty))$
.53 $(a \in \mathrm{rl} \wedge b \in \mathrm{rl} \to a < b \vee a = b \vee b < a)$
.54 $(a \in \mathrm{rl} \wedge b \in \mathrm{rl} \to a < b \vee b \backsim\leq a)$
.55 $(0 < 1)$
.56 $(x < y \wedge 0 < z < \infty \to x \cdot z < y \cdot z)$
.57 $(x \in \mathrm{rl} \wedge 0 < z < \infty \to x \cdot z \in \mathrm{rl})$
.58 $(x \leq y \wedge 0 < z < \infty \to x \cdot z \leq y \cdot z)$
.59 $(-\infty < x \leq y < \infty \wedge 0 \leq z < \infty \to x \cdot z \leq y \cdot z)$
.60 $(0 \leq a \leq b < \infty \wedge 0 \leq c \leq c < \infty \to 0 \leq a \cdot c \leq b \cdot d < \infty)$
.61 $(0 \leq a < b \wedge 0 \leq c < d \to 0 \leq a \cdot c < b \cdot d)$
.62 $(b \in \mathrm{rl} \wedge z \in \mathrm{rf} \to b + z \in \mathrm{rl})$
.63 $(x < y \wedge z \in \mathrm{rf} \to x + z < y + z)$

.64 $(x \leq y \wedge z \in \mathrm{rf} \rightarrow x + z \leq y + z)$
.65 $(a < b \wedge c < d \rightarrow a + c < b + d)$
.66 $(0 \leq x \wedge 0 \leq y \rightarrow 0 \leq x + y)$
.67 $((0 < x \leftrightarrow -x < 0) \wedge (0 \leq x \leftrightarrow -x \leq 0))$
.68 $(x \in \mathrm{rl} \wedge y \in \mathrm{rl} \rightarrow$
$(x + y < \infty \leftrightarrow x < \infty \wedge y < \infty) \wedge$
$(-\infty < x + y \leftrightarrow -\infty < x \wedge -\infty < y) \wedge$
$(x + y \in \mathrm{rf} \leftrightarrow x \in \mathrm{rf} \wedge y \in \mathrm{rf}))$
.69 $(0 \leq b \rightarrow x + b \backsim< x)$
.70 $(-\infty < x \rightarrow x \leq y \leftrightarrow \bigvee h \geq 0(y = x + h))$
.71 $(-\infty < a \leq b \wedge -\infty < c \leq d \rightarrow a + c \leq b + d)$
.72 $(a \leq b < \infty \wedge c \leq d < \infty \rightarrow a + c \leq b + d)$
.73 $(a < b \rightarrow \bigvee c(a < c < b))$
.74 $(A \subset \mathrm{rl} \wedge y \in A \rightarrow y \leq \mathrm{Sup}\, A)$
.75 $(A \subset \mathrm{rl} \wedge t \in \mathrm{rl} \wedge \bigwedge y \in A(y \leq t) \rightarrow \mathrm{Sup}\, A \leq t)$
.76 $(t < \mathrm{Sup}\, A \rightarrow \bigvee y(t < y \in A))$
.77 $(x \in \mathrm{rl} \rightarrow \mathrm{Sup}\,\mathrm{sng}\, x = x)$
.78 $(\mathrm{Sup}\, 0 = -\infty)$
.79 $(\mathrm{Sup}\, \omega = \infty)$
.80 $(\infty \in \mathrm{rp})$
.81 $(\infty \cdot \infty = \infty)$
.82 $(0 < k \leftrightarrow k \cdot \infty = \infty)$
.83 $(x \in \mathrm{rl} \wedge x \cdot y \in \mathrm{rl} \rightarrow y \in \mathrm{rl})$
.84 $(x \in \mathrm{rl} \wedge x \cdot y \in \mathrm{spl} \rightarrow y \in \mathrm{spl})$
.85 $(x \in \mathrm{infin} \wedge y \in \mathrm{infin} \wedge x \neq y \rightarrow x + y = \mathrm{U})$
.86 $(0 \neq x \in \mathrm{kf} \cup \mathrm{dinfin} \wedge y \in \mathrm{dinfin} \rightarrow x \cdot y \in \mathrm{dinfin})$
.87 $(0 \neq x \in \mathrm{kt} \wedge y \in \mathrm{infin} \rightarrow x \cdot y \in \mathrm{infin})$
.88 $(a \in \mathrm{kt} \wedge b \in \mathrm{kt} \rightarrow 1/(a \cdot b) = (1/a) \cdot (1/b))$
.89 $(x \in \mathrm{cp} \leftrightarrow -x \in \mathrm{cp} \leftrightarrow 1/x \in \mathrm{cp})$
.90 $(x + y \neq \mathrm{U} \rightarrow x \in \mathrm{cp} \wedge y \in \mathrm{cp} \leftrightarrow x + y \in \mathrm{cp})$
.91 $(x + y \neq \mathrm{U} \rightarrow x \in \mathrm{spl} \wedge y \in \mathrm{spl} \leftrightarrow x + y \in \mathrm{spl})$
.92 $(x + y \neq \mathrm{U} \wedge x \in \mathrm{rl} \wedge y \in \mathrm{rl} \rightarrow x + y \in \mathrm{rl})$
.93 $(x \cdot y \neq \mathrm{U} \wedge x \in \mathrm{cp} \wedge y \in \mathrm{cp} \rightarrow x \cdot y \in \mathrm{cp})$
.94 $(x \cdot y \neq \mathrm{U} \wedge x \in \mathrm{rl} \wedge y \in \mathrm{rl} \rightarrow x \cdot y \in \mathrm{rl})$

Definition
1.12 $(\surd x \equiv (0 \leq x \rightarrow \mathrm{Sup}\, \sqsubseteq t \geq 0(\dot{}t2 \leq x)))$

Theorem
1.13 $(0 \leq x \rightarrow 0 \leq \surd x \wedge \dot{}\surd x2 = x = \surd\dot{}x2)$

1.14 Definitions
.0 $(\mathrm{prt}'\, x \equiv \mathrm{The}\, a \in \mathrm{rl} \bigvee b \in \mathrm{rf}(x = a + \mathrm{i} \cdot b))$
.1 $(\mathrm{prt}''\, x \equiv (x \backsim\in \mathrm{rl} \wedge \mathrm{The}\, b \in \mathrm{rf} \bigvee a \in \mathrm{rf}(x = a + \mathrm{i} \cdot b))$
.2 $(|x| \equiv (x \in \mathrm{kf} \wedge \surd(\dot{}\,\mathrm{prt}'\, x2 + \dot{}\,\mathrm{prt}''\, x2) \vee x \backsim\in \mathrm{kf} \wedge \infty))$

1.15 Theorems

.0 $(x \in \mathrm{spl} \leftrightarrow \mathrm{prt}'\, x \neq \mathrm{U} \leftrightarrow \mathrm{prt}''\, x \neq \mathrm{U} \leftrightarrow \mathrm{prt}'\, x \in \mathrm{rl} \leftrightarrow \mathrm{prt}''\, x \in \mathrm{rf})$

.1 $(x \in \mathrm{kf} \leftrightarrow \mathrm{prt}'\, x \in \mathrm{rf})$

.2 $(x \in \mathrm{rl} \leftrightarrow \mathrm{prt}''\, x = 0)$

.3 $(x \in \mathrm{rl} \rightarrow x = \mathrm{prt}'\, x)$

.4 $(x \in \mathrm{spl} \rightarrow x = \mathrm{prt}'\, x + \mathrm{i} \cdot \mathrm{prt}''\, x)$

.5 $(a \in \mathrm{rf} \wedge b \in \mathrm{rf} \wedge x = a + \mathrm{i} \cdot b \rightarrow a = \mathrm{prt}'\, x \wedge b = \mathrm{prt}''\, x)$

.6 $(\mathrm{prt}'(x + y) = \mathrm{prt}'\, x + \mathrm{prt}'\, y)$

.7 $(x + y \in \mathrm{kf} \rightarrow \mathrm{prt}''(x + y) = \mathrm{prt}''\, x + \mathrm{prt}''\, y)$

.8 $(c \in \mathrm{rl} \wedge c \cdot x \in \mathrm{spl} \rightarrow x \in \mathrm{spl})$

.9 $(0 \neq c \in \mathrm{rf} \rightarrow c \cdot x \in \mathrm{spl} \leftrightarrow x \in \mathrm{spl})$

.10 $(0 \neq c \in \mathrm{rf} \rightarrow \mathrm{prt}'(c \cdot x) = c \cdot \mathrm{prt}'\, x)$

1.16 Theorems

.0 $(0 \leq |x| \leq \infty)$

.1 $(x \in \mathrm{kf} \leftrightarrow 0 < |x| < \infty)$

.2 $(|x| = 0 \leftrightarrow x = 0)$

.3 $(|-x| = |x|)$

.4 $(|x + y| \leq |x| + |y|)$

.5 $(||x| - |y|| \leq |x - y|)$

.6 $(|x| \cdot |y| \neq \mathrm{U} \rightarrow |x \cdot y| = |x| \cdot |y|)$

.7 $(0 \neq y \in \mathrm{kt} \rightarrow |1/y| = 1/|y|)$

.8 $(x \in \mathrm{rl} \rightarrow |x| = x \vee |x| = -x)$

.9 $(x \in \mathrm{rl} \rightarrow x \leq |x|)$

.10 $(|\mathrm{prt}'\, x| \leq |x| \wedge |\mathrm{prt}''\, x| \leq |x|)$

.11 $(|x| \leq |\mathrm{prt}'\, x| + |\mathrm{prt}''\, x|)$

1.17 Definitions

.0 $(\text{transitive is } R \equiv (\text{relation is } R \wedge \bigwedge x \bigwedge y \bigwedge z(x, y \in R \wedge y, z \in R \rightarrow x, z \in R)))$

.1 $(\text{direction is } R \equiv (R \neq 0 \wedge \text{transitive is } R \wedge$
$\bigwedge x \in \mathrm{dmn}\, R \bigwedge y \in \mathrm{dmn}\, R \bigvee z \in \mathrm{dmn}\, R(x, z \in R \wedge y, z \in R)))$

.2 $(\text{run is } R \equiv (\text{relation is } R \wedge R \neq 0 \wedge$
$\bigwedge x \in \mathrm{dmn}\, R \bigwedge y \in \mathrm{dmn}\, R \bigvee z \in \mathrm{dmn}\, R(\mathrm{vs}\, Rz \subset \mathrm{vs}\, Rx \cap \mathrm{vs}\, Ry)))$

.3 $(\text{far } Rx\underline{\mathrm{u}}x \equiv (\text{run is } R \wedge \bigvee \delta \in \mathrm{dmn}\, R \bigwedge x \in \mathrm{vs}\, R\delta\underline{\mathrm{u}}x))$

Theorem

1.18 $(\text{relation is } R \rightarrow \text{transitive is } R \leftrightarrow R : R \subset R)$

Theorem

1.19 $(\text{direction is } R \rightarrow \text{run is } R)$

Theorem

1.20 $(\text{far } Rx\underline{\mathrm{u}}x \wedge \text{far } Rx\underline{\mathrm{v}}x \leftrightarrow \text{far } Rx(\underline{\mathrm{u}}x \wedge \underline{\mathrm{v}}x))$

1.21 Definitions

.0 $(\mathrm{Nb}\, ra \equiv (a = -\infty \wedge \exists\, x(\mathrm{prt}'\, x \leq -1/r) \vee$
$a \in \mathrm{kf} \wedge \exists\, x(|x - a| \leq r) \vee$
$a = \infty \wedge \exists\, x(\mathrm{prt}'\, x \geq 1/r)))$

.1 $(\mathrm{Nb}'\, ra \equiv (\mathrm{Nb}\, ra {\sim} \mathrm{sng}\, a))$

1.22 Definitions

.0 $(\mathrm{disc}\, ra \equiv (a \in \mathrm{kf} \wedge \mathrm{cp}\, \mathrm{Nb}\, ra \vee a = \infty\kern-0.6em\mid\; \wedge \exists\, x \in \mathrm{cp}(1/x \in \mathrm{Nb}\, r0)))$

.1 $(\mathrm{disc}'\, ra \equiv (\mathrm{disc}\, ra {\sim} \mathrm{sng}\, a))$

1.23 Definitions

.0 $((\underline{\mathrm{u}}x \text{ tends to } p \text{ as } x \text{ runs along } R) \equiv \bigwedge r \in \mathrm{rfp}\; \mathrm{far}\; Rx(\underline{\mathrm{u}}x \in \mathrm{Nb}\, rp))$

.1 $((\underline{\mathrm{u}}x \text{ slides to } p \text{ as } x \text{ runs along } R) \equiv \bigwedge r \in \mathrm{rfp}\; \mathrm{far}\; Rx(\underline{\mathrm{u}}x \in \mathrm{disc}\, rp))$

.2 $((\underline{\mathrm{u}}x \rightarrow p \text{ as } x \cdot\!\!\rightarrow R) \equiv (\underline{\mathrm{u}}x \text{ tends to } p \text{ as } x \text{ runs along } R))$

.3 $((\underline{\mathrm{u}}x \hookrightarrow p \text{ as } x \cdot\!\!\rightarrow R) \equiv (\underline{\mathrm{u}}x \text{ slides to } p \text{ as } x \text{ runs along } R))$

.4 $(\mathrm{lm}\, xR\underline{\mathrm{u}}x \equiv \mathrm{The}\, p(\underline{\mathrm{u}}x \rightarrow p \text{ as } x \cdot\!\!\rightarrow R))$

.5 $(\mathrm{Lm}\, xR\underline{\mathrm{u}}x \equiv \mathrm{The}\, p(\underline{\mathrm{u}}x \hookrightarrow p \text{ as } x \cdot\!\!\rightarrow R))$

.6 $(\mathrm{lmrn}\, Ca \equiv \exists\, r, y(0 < r < \infty \wedge y \in C\, \mathrm{Nb}'\, ra))$

.7 $(\mathrm{Lmrn}\, Ca \equiv \exists\, r, y(0 < r < \infty \wedge y \in C\, \mathrm{disc}'\, ra))$

.8 $(\lim C \ni xa\underline{\mathrm{u}}x \equiv \mathrm{lm}\, x\, \mathrm{lmrn}\, Ca\underline{\mathrm{u}}x)$

.9 $(\mathrm{Lim}\, C \ni xa\underline{\mathrm{u}}x \equiv \mathrm{Lm}\, x\, \mathrm{Lmrn}\, Ca\underline{\mathrm{u}}x)$

.10 $(\lim xa\underline{\mathrm{u}}x \equiv \lim \mathrm{rl} \ni xa\underline{\mathrm{u}}x)$

.11 $(\mathrm{Lim}\, xa\underline{\mathrm{u}}x \equiv \mathrm{Lim}\, \mathrm{cp} \ni xa\underline{\mathrm{u}}x)$

Theorem

1.24 $((\underline{\mathrm{u}}x \rightarrow p \text{ as } x \cdot\!\!\rightarrow R) \wedge (\underline{\mathrm{u}}x \rightarrow q \text{ as } x \cdot\!\!\rightarrow R) \rightarrow p = q \in \mathrm{spl})$

Theorem

1.25 $((\underline{\mathrm{u}}x \hookrightarrow p \text{ as } x \cdot\!\!\rightarrow R) \wedge (\underline{\mathrm{u}}x \hookrightarrow q \text{ as } x \cdot\!\!\rightarrow R) \rightarrow p = q \in \mathrm{cp})$

1.26 Theorems

.0 $((\underline{\mathrm{u}}x \rightarrow p \text{ as } x \cdot\!\!\rightarrow R) \leftrightarrow \mathrm{lm}\, xR\underline{\mathrm{u}}x = p \neq \mathrm{U})$

.1 $((\underline{\mathrm{u}}x \hookrightarrow p \text{ as } x \cdot\!\!\rightarrow R) \leftrightarrow \mathrm{Lm}\, xR\underline{\mathrm{u}}x = p \neq \mathrm{U})$

1.27 Theorems

.0 $(a \in \mathrm{kf} \rightarrow \mathrm{lm}\, xR\underline{\mathrm{u}}x = a \leftrightarrow \mathrm{Lm}\, xR\underline{\mathrm{u}}x = a)$

.1 $(a \in \mathrm{kf} \wedge b \in \mathrm{kf} \rightarrow \lim \mathrm{kf} \ni xa\underline{\mathrm{u}}x = b \leftrightarrow \mathrm{Lim}\, xa\underline{\mathrm{u}}x = b)$

Theorem

1.28 $(\mathrm{far}\, Rx(\underline{\mathrm{u}}x = \underline{\mathrm{v}}x) \rightarrow \mathrm{lm}\, xR\underline{\mathrm{u}}x = \mathrm{lm}\, xR\underline{\mathrm{v}}x \wedge \mathrm{Lm}\, xR\underline{\mathrm{u}}x = \mathrm{Lm}\, xR\underline{\mathrm{v}}x)$

Theorem

1.29 $(\mathrm{run\, is}\, R \wedge a \in \mathrm{spl} \rightarrow \mathrm{lm}\, xRa = a)$

Theorem

1.30 $(\mathrm{lm}\, xR\underline{\mathrm{u}}x = A \in \mathrm{kf} \wedge \mathrm{lm}\, xR\underline{\mathrm{v}}x = B \in \mathrm{kf} \rightarrow$

.0 $\mathrm{lm}\, xR(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) = A + B \wedge$

.1 $\mathrm{lm}\, xR(\underline{\mathrm{u}}x \cdot \underline{\mathrm{v}}x) = A \cdot B)$

Theorem
1.31 $(0 \neq \operatorname{lm} xR\underline{\mathrm{u}}x = A \in \mathrm{kf} \to \operatorname{lm} xR(1/\underline{\mathrm{u}}x) = 1/A)$

Theorem
1.32 $(\operatorname{lm} xR\underline{\mathrm{u}}x = A \in \mathrm{kf} \wedge k \in \mathrm{kf} \to \lim xR(k \cdot \underline{\mathrm{u}}x) = k \cdot A)$

Theorem
1.33 $(\operatorname{lm} xR\underline{\mathrm{u}}x = A \in \mathrm{kf} \to$
.0 $\operatorname{lm} xR \operatorname{prt}' \underline{\mathrm{u}}x = \operatorname{prt}' A \wedge$
.1 $\operatorname{lm} xR \operatorname{prt}'' \underline{\mathrm{u}}x = \operatorname{prt}'' A)$

Theorem
1.34 $(\operatorname{lm} xR\underline{\mathrm{u}}x = A \in \mathrm{kf} \to \operatorname{lm} xR|x| = |A|)$

Lemma
1.35 $(r \in \mathrm{rfp} \wedge a \in \mathrm{rl} \cap \mathrm{infin} \to \underline{\mathrm{u}}x \in \mathrm{Nb}\, ra \leftrightarrow \operatorname{prt}' \underline{\mathrm{u}}x \in \mathrm{Nb}\, ra)$

Theorem
1.36 $(a \in \mathrm{rl} \cap \mathrm{infin} \to \operatorname{lm} xR\underline{\mathrm{u}}x = a \leftrightarrow \operatorname{lm} xR \operatorname{prt}' \underline{\mathrm{u}}x = a)$

Theorem
1.37 $(\operatorname{lm} xR\underline{\mathrm{u}}x \neq \mathrm{U} \to \operatorname{prt}' \operatorname{lm} xR\underline{\mathrm{u}}x = \operatorname{lm} xR \operatorname{prt}' \underline{\mathrm{u}}x)$

Theorem
1.38 $(0 \neq k \in \mathrm{kf} \to \operatorname{lm} xR(k \cdot \underline{\mathrm{u}}x) = k \cdot \operatorname{lm} xR\underline{\mathrm{u}}x)$

Theorem
1.39 $(\operatorname{far} Rx(|\underline{\mathrm{u}}x| \leq \underline{\mathrm{v}}x) \wedge \operatorname{lm} xR\underline{\mathrm{v}}x = 0 \to \operatorname{lm} xR\underline{\mathrm{u}}x = 0)$

Theorem
1.40 $(\operatorname{far} Rx(\underline{\mathrm{u}}x \in \mathrm{rl}) \to \operatorname{lm} xR\underline{\mathrm{u}}x = \infty \leftrightarrow \operatorname{far} Rx(\underline{\mathrm{u}}x > 0) \wedge \operatorname{lm} xR(1/\underline{\mathrm{u}}x) = 0)$

1.41 Theorems
.0 $(\operatorname{far} Rx(\underline{\mathrm{u}}x \leq \underline{\mathrm{v}}x) \wedge \operatorname{lm} xR\underline{\mathrm{u}}x = \infty \to \operatorname{lm} xR\underline{\mathrm{v}}x = \infty)$
.1 $(\operatorname{far} Rx(\underline{\mathrm{u}}x \leq \underline{\mathrm{v}}x) \wedge \operatorname{lm} xR\underline{\mathrm{v}}x = -\infty \to \operatorname{lm} xR\underline{\mathrm{u}}x = -\infty)$

Lemma
1.42 $(M \in \mathrm{rfp} \wedge \operatorname{far} Rx(\underline{\mathrm{u}}x \geq -M) \wedge \operatorname{far} Rx(\underline{\mathrm{v}}x \in \mathrm{rl}) \wedge \operatorname{lm} xR\underline{\mathrm{v}}x = \infty \to \operatorname{lm} xR(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) = \infty)$

Lemma
1.43 $(M \in \mathrm{rfp} \wedge \operatorname{far} Rx(\underline{\mathrm{u}}x > M) \wedge \operatorname{far} Rx(\underline{\mathrm{v}}x \in \mathrm{rl}) \wedge \operatorname{lm} xR\underline{\mathrm{v}}x = \infty \to \operatorname{lm} xR(\underline{\mathrm{u}}x \cdot \underline{\mathrm{v}}x) = \infty)$

Theorem
1.44 $(\mathrm{far}\, Rx(\underline{\mathrm{u}}x \in \mathrm{rl} \wedge \underline{\mathrm{v}}x \in \mathrm{rl}) \wedge \mathrm{lm}\, xR\underline{\mathrm{u}}x = A \wedge \mathrm{lm}\, xR\underline{\mathrm{v}}x = B \rightarrow$
.0 $(A + B \neq \mathrm{U} \rightarrow \mathrm{lm}\, xR(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) = A + B) \wedge$
.1 $(A \cdot B \neq \mathrm{U} \rightarrow \mathrm{lm}\, xR(\underline{\mathrm{u}}x \cdot \underline{\mathrm{v}}x) = A \cdot B) \wedge$
.2 $(0 \neq A \neq \mathrm{U} \rightarrow \mathrm{lm}\, xR(1/\underline{\mathrm{u}}x) = 1/A) \wedge$
.3 $(A \neq \mathrm{U} \rightarrow A \in \mathrm{rl}) \wedge$
.4 $(\mathrm{far}\, Rx(\underline{\mathrm{u}}x \leq \underline{\mathrm{v}}x) \rightarrow B \backsim\!\leq A))$

Theorem
1.45 $(\mathrm{far}\, Rx(\underline{\mathrm{u}}x > 0) \wedge \mathrm{lm}\, xR\underline{\mathrm{u}}x = 0 \rightarrow \mathrm{lm}\, xR(1/\underline{\mathrm{u}}x) = \infty)$

1.46 Theorems
.0 $(\mathrm{lm}\, xR\underline{\mathrm{u}}x + \mathrm{lm}\, xR\underline{\mathrm{v}}x = A \neq \mathrm{U} \rightarrow \mathrm{lm}\, xR(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) = A)$
.1 $(\mathrm{lm}\, xR\underline{\mathrm{u}}x = A \neq \mathrm{U} \rightarrow \mathrm{lm}\, xR|\underline{\mathrm{u}}x| = |A|)$

Theorem
1.47 $(\mathrm{run\, is}\, R \wedge a \in \mathrm{cp} \rightarrow \mathrm{Lm}\, xRa = a)$

Theorem
1.48 $(\mathrm{far}\, Rx(\underline{\mathrm{u}}x \in \mathrm{cp}) \rightarrow \mathrm{Lm}\, xR(1/\underline{\mathrm{u}}x) = 1/\, \mathrm{Lm}\, xR\underline{\mathrm{u}}x)$

Theorem
1.49 $(\mathrm{Lm}\, xR\underline{\mathrm{u}}x = A \wedge \mathrm{Lm}\, xR\underline{\mathrm{v}}x = B \rightarrow$
.0 $(A + B \neq \mathrm{U} \rightarrow \mathrm{Lm}\, xR(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) = A + B) \wedge$
.1 $(A \cdot B \neq \mathrm{U} \rightarrow \mathrm{Lm}\, xR(\underline{\mathrm{u}}x \cdot \underline{\mathrm{v}}x) = A \cdot B) \wedge$
.2 $(0 \neq z \in \mathrm{kf} \rightarrow \mathrm{Lm}\, xR(z \cdot \underline{\mathrm{u}}x) = z \cdot A))$

Theorem
1.50 $(\mathrm{far}\, Rx(|\underline{\mathrm{u}}x| \leq M < \infty) \rightarrow \mathrm{lm}\, xR\underline{\mathrm{u}}x = \mathrm{lm}\, xR\, \mathrm{prt}'\, \underline{\mathrm{u}}x + \mathrm{i} \cdot \mathrm{lm}\, xR\, \mathrm{prt}''\, \underline{\mathrm{u}}x)$

1.51 Theorems
.0 $(\mathrm{run\, is}\, R \wedge N \in \mathrm{dmn}\, R \wedge \bigwedge y \in \mathrm{vs}\, RN\, \mathrm{far}\, Rx(\centerdot fy \leq \centerdot fx) \rightarrow \mathrm{lm}\, xR \centerdot fx = \sup x \in \mathrm{vs}\, RN \centerdot fx)$
.1 $(\mathrm{run\, is}\, R \wedge N \in \mathrm{dmn}\, R \wedge \bigwedge y \in \mathrm{vs}\, RN\, \mathrm{far}\, Rx(\centerdot fy \geq \centerdot fx) \rightarrow \mathrm{lm}\, xR \centerdot fx = \inf x \in \mathrm{vs}\, RN \centerdot fx)$

1.52 Theorems
.0 $(\mathrm{far}\, Ry\, \mathrm{far}\, Rx(\centerdot fy \leq \centerdot fx) \rightarrow -\infty \leq \mathrm{lm}\, xR \centerdot fx \leq \infty)$
.1 $(\mathrm{far}\, Ry\, \mathrm{far}\, Rx(\centerdot fy \geq \centerdot fx) \rightarrow -\infty \leq \mathrm{lm}\, xR \centerdot fx \leq \infty)$

In 1.52 above we find a generalization of the fact that monotone functions have limits.

1.53 Definitions
.0 $(\mathrm{lin}\, n\underline{\mathrm{u}}n \equiv \mathrm{lim}\, \omega \ni n\infty\underline{\mathrm{u}}n)$
.1 $(\mathrm{Lin}\, n\underline{\mathrm{u}}n \equiv \mathrm{Lim}\, \omega \ni n\infty\!\!\!\!\mid\;\underline{\mathrm{u}}n)$
.2 $(\mathrm{large}\, x \in A\underline{\mathrm{u}}x \equiv \bigvee y \in \mathrm{rfp} \bigwedge x \in \mathrm{rfp}(y \leq x \in A \rightarrow \underline{\mathrm{u}}x))$
.3 $(\mathrm{small}\, x \in A\underline{\mathrm{u}}x \equiv \bigvee y \in \mathrm{rfp} \bigwedge x \in \mathrm{rfp}(y \geq x \in A \rightarrow \underline{\mathrm{u}}x))$
.4 $(\mathrm{large}\, x\underline{\mathrm{u}}x \equiv \mathrm{large}\, x \in \mathrm{rfp}\, \underline{\mathrm{u}}x)$
.5 $(\mathrm{small}\, x\underline{\mathrm{u}}x \equiv \mathrm{small}\, x \in \mathrm{rfp}\, \underline{\mathrm{u}}x)$
.6 $(\mathrm{big}\, n\underline{\mathrm{u}}n \equiv \mathrm{large}\, n \in \omega\underline{\mathrm{u}}n)$

Definition

1.54 (indexrun $R \equiv \sqsubseteq \xi, \eta, \in \operatorname{dmn} R(\operatorname{run} \operatorname{is} R \wedge \operatorname{vs} R\xi \supset \operatorname{vs} R\eta)$)

1.55 Theorems

.0 ($\operatorname{run} \operatorname{is} R \wedge I = \operatorname{indexrun} R \rightarrow \operatorname{dmn} I = \operatorname{rng} I = \operatorname{dmn} R \wedge \bigwedge \xi \in \operatorname{dmn} I(\xi \in \operatorname{vs} I\xi)$)

.1 ($\operatorname{direction} \operatorname{is} \operatorname{indexrun} R \leftrightarrow \operatorname{run} \operatorname{is} R$)

.2 ($\operatorname{direction} \operatorname{is} R \wedge \bigwedge \xi \in \operatorname{dmn} R(\xi \in \operatorname{vs} R\xi) \rightarrow \operatorname{indexrun} R = \operatorname{sqr} \operatorname{dmn} R \cap R$)

.3 ($\operatorname{direction} \operatorname{is} R \wedge \bigwedge \xi \in \operatorname{dmn} R(\xi \in \operatorname{vs} R\xi) \wedge \operatorname{dmn} R = \operatorname{rng} R \rightarrow \operatorname{indexrun} R = R$)

.4 ($\operatorname{run} \operatorname{is} R \wedge \operatorname{dmn}(S : R) = \operatorname{dmn} R \rightarrow \operatorname{run} \operatorname{is}(S : R)$)

.5 ($\operatorname{run} \operatorname{is} R \wedge \operatorname{function} \operatorname{is} f \wedge \operatorname{rng} R \subset \operatorname{rng} f \wedge R' = \sqsubseteq \xi, y(\centerdot\, fy \in \operatorname{vs} R\xi) \rightarrow R' = \operatorname{inv} f : R$)

1.56 Definitions

.0 ($\overline{\operatorname{lm}}\, xR\underline{\operatorname{u}}x \equiv \operatorname{lm} \xi \operatorname{indexrun} R \sup x \in \operatorname{vs} R\xi\underline{\operatorname{u}}x$)

.1 ($\underline{\operatorname{lm}}\, xR\underline{\operatorname{u}}x \equiv \operatorname{lm} \xi \operatorname{indexrun} R \inf x \in \operatorname{vs} R\xi\underline{\operatorname{u}}x$)

.2 ($\overline{\operatorname{lim}}\, xa\underline{\operatorname{u}}x \equiv \overline{\operatorname{lm}}\, x \operatorname{lmrn} \operatorname{rl} a\underline{\operatorname{u}}x$)

.3 ($\underline{\operatorname{lim}}\, xa\underline{\operatorname{u}}x \equiv \underline{\operatorname{lm}}\, x \operatorname{lmrn} \operatorname{rl} a\underline{\operatorname{u}}x$)

.4 ($\overline{\operatorname{lin}}\, n\underline{\operatorname{u}}n \equiv \overline{\operatorname{lm}}\, x \operatorname{lmrn} \omega\infty\underline{\operatorname{u}}n$)

.5 ($\underline{\operatorname{lin}}\, n\underline{\operatorname{u}}n \equiv \underline{\operatorname{lm}}\, x \operatorname{lmrn} \omega\infty\underline{\operatorname{u}}n$)

1.57 Suggestion

Formulate some properties of $\underline{\operatorname{lm}}$ and $\overline{\operatorname{lm}}$ and incorporate them in Theorems.

Finite Summation

1.58 Definitions

.0 $((a \text{ eq } b) \equiv \bigvee f(\text{univalent is } f \wedge \text{dmn } f = a \wedge \text{rng } f = b))$

.1 $(\text{pwr}' A \equiv \text{Inf} \mathrel{E} n(A \text{ eq } n \in \omega))$

1.59 Definitions

.0 $(\text{summ } \varphi f A \equiv (\varphi \in \text{Upon sb } A \text{ To kt} \wedge f \in \text{To kt} \wedge \text{pwr}' A \in \omega \wedge .\varphi 0 = 0 \wedge \bigwedge x \in A(.\varphi \text{ sng } x = .fx) \wedge \bigwedge T \subset A \bigwedge B(.\varphi T = .\varphi(TB) + .\varphi(T{\sim}B))))$

.1 $(\text{sm } fA \equiv \mathrel{E} \varphi \text{ summ } \varphi f A)$

.2 $(\text{sum } fA \equiv . \nabla \text{ sm } fAA)$

.3 $(\text{ad } x \in A\underline{u}x \equiv \text{sum} \curlywedge x \in A\underline{u}xA)$

Theorem

1.60 $(\text{pwr}' A \in \omega \wedge f \in \text{To kt} \rightarrow \text{singleton is sm } fA)$

Proof:

Letting

$(N = \mathrel{E} n \in \omega \bigwedge \alpha(\text{pwr}' \alpha = n \rightarrow \text{singleton } is \text{ sm } f\alpha))$

we complete the proof by verifying the

Statement

$(\omega{\sim}N = 0)$

Proof (by contradiction):

Suppose

$(\omega{\sim}N \neq 0)$,

let

$(n_0 = \text{Inf}(\omega{\sim}N))$,

and so select A_0 that

$(\text{pwr}' A_0 = n_0)$

and

.0 $\sim \text{singleton is sm } fA_0$.

We readily see that

$(n_0 > 1)$

and secure such a C that

$(A_0C \neq 0 \wedge A_0{\sim}C \neq 0)$.

Since evidently

$(\text{pwr}'(A_0C) < n_0 \wedge \text{pwr}'(A_0{\sim}C) < n_0)$

we are sure that

$(\text{singleton is sm } f(A_0C) \wedge \text{singleton is sm } f(A_0{\sim}C))$.

Let

$(\varphi_1 = \nabla \text{ sm } f(A_0C) \wedge \varphi_2 = \nabla \text{ sm } f(A_0{\sim}C))$

and notice that

$(\text{summ } \varphi_1 f(A_0C) \wedge \text{summ } \varphi_2 f(A_0{\sim}C))$.

Take

.1 $(\varphi_0 = \curlywedge \beta \subset A_0(.\varphi_1(\beta C) + .\varphi_2(\beta{\sim}C)))$

and observe that

$\text{summ } \varphi_0 f A_0$

and hence that

.2 $(\varphi_0 \in \text{sm } fA_0)$.

Now let ψ_0 be any member of $\mathrm{sm}\, f A_0$ and let

$(\psi_1 = \curlywedge\beta \subset A_0 \cap C \,\boldsymbol{.}\, \psi_0\beta \wedge \psi_2 = \curlywedge\beta \subset A_0 \backsim C \,\boldsymbol{.}\, \psi_0\beta)$.

Check that

$(\psi_1 \in \mathrm{sm}\, f(A_0C) \wedge \psi_2 \in \mathrm{sm}\, f(A_0{\sim}C))$.

Infer next that the singleton nature of of $\mathrm{sm}\, f(A_0C)$ and $\mathrm{sm}\, f(A_0{\sim}C)$ requires that

$(\psi_1 = \varphi_1 \wedge \psi_2 = \varphi_2)$.

Consequently, in view of .1, it is seen that

$$\begin{aligned}(T \subset A_0 \rightarrow \boldsymbol{.}\, \psi_0 T\\ &= \boldsymbol{.}\, \psi_0(TC) + \boldsymbol{.}\, \psi_0(T{\sim}C)\\ &= \boldsymbol{.}\, \psi_1(TC) + \boldsymbol{.}\, \psi_2(T{\sim}C)\\ &= \boldsymbol{.}\, \varphi_1(TC) + \boldsymbol{.}\, \varphi_2(T{\sim}C)\\ &= \boldsymbol{.}\, \varphi_0 T)\ .\end{aligned}$$

Accordingly

$(\psi_0 = \varphi_0)$.

From this and .2 we conclude, in contradiction to .0, that

singleton is $\mathrm{sm}\, f A_0$.

1.61 Lemmas

.0 $(f \in \mathrm{To\,kt} \wedge A \in \mathrm{fnt} \wedge \varphi \in \mathrm{sm}\, fA \rightarrow \mathrm{sum}\, fa = \boldsymbol{.}\, \varphi A)$

.1 $(f \in \mathrm{To\,kt} \wedge A \in \mathrm{fnt} \wedge \varphi \in \mathrm{sm}\, fA \wedge B \subset A \rightarrow \mathrm{sum}\, fB = \boldsymbol{.}\, \varphi B)$

1.62 Theorems

.0 $(f \backsim\in \mathrm{To\,kt} \vee A \backsim\in \mathrm{fnt} \rightarrow \mathrm{sum}\, fA = \mathrm{U})$

.1 $(\mathrm{sum}\, fT = \mathrm{sum}\, f(TB) + \mathrm{sum}\, f(T{\sim}B))$

.2 $(A \cap B = 0 \rightarrow \mathrm{sum}\, fA + \mathrm{sum}\, fB = \mathrm{sum}\, f(A \cup B))$

.3 $(f \in \mathrm{To\,kt} \rightarrow \mathrm{sum}\, f\, \mathrm{sngl}\, y = \boldsymbol{.}\, fy)$

.4 $(f \in \mathrm{To\,kt} \rightarrow \mathrm{sum}\, f0 = 0)$

1.63 Theorems

.0 $(A \backsim\in \mathrm{fnt} \rightarrow \mathrm{ad}\, x \in A\underline{\mathrm{u}}x = \mathrm{U})$

.1 $(\bigwedge x \in A(\underline{\mathrm{u}}x = \underline{\mathrm{v}}x) \rightarrow \mathrm{ad}\, x \in A\underline{\mathrm{u}}x = \mathrm{ad}\, x \in A\underline{\mathrm{v}}x)$

.2 $(f \in \mathrm{To\,kt} \rightarrow \mathrm{ad}\, x \in A\underline{\mathrm{u}}x = \mathrm{sum}\, fA)$

.3 $(A \cap B = 0 \rightarrow \mathrm{ad}\, x \in A\underline{\mathrm{u}}x + \mathrm{ad}\, x \in B\underline{\mathrm{u}}x = \mathrm{ad}\, x \in A \cup B\underline{\mathrm{u}}x)$

.4 $(y \in \mathrm{U} \wedge \underline{\mathrm{u}}y \in \mathrm{kt} \rightarrow \mathrm{ad}\, x \in \mathrm{sng}\, y\underline{\mathrm{u}}x = \underline{\mathrm{u}}y)$

.5 $(y \in \mathrm{U} \wedge \underline{\mathrm{u}}y \backsim\in \mathrm{kt} \rightarrow \mathrm{ad}\, x \in \mathrm{sng}\, y\underline{\mathrm{u}}x = \mathrm{U})$

.6 $(\mathrm{ad}\, x \in 0\underline{\mathrm{u}}x = 0)$

.7 $(\mathrm{ad}\, x \in A(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) = \mathrm{ad}\, x \in A\underline{\mathrm{u}}x + \mathrm{ad}\, x \in A\underline{\mathrm{v}}x)$

Hint: Induction in $\mathrm{pwr}'\, A$. Check carefully when $(\mathrm{pwr}'\, A = 1)$.

.8 $(c \in \mathrm{kf} \rightarrow \mathrm{ad}\, x \in A(c \cdot \underline{\mathrm{u}}x) = c \cdot \mathrm{ad}\, x \in A\underline{\mathrm{u}}x)$

.9 $(\mathrm{ad}\, x \in A\underline{\mathrm{u}}x \neq \mathrm{U} \rightarrow \mathrm{ad}\, x \in A\underline{\mathrm{u}}x \in \mathrm{kt})$

Theorems

1.63 $(A \in \mathrm{fnt} \rightarrow$

.0 $(\bigwedge x \in A(\underline{\mathrm{u}}x < \infty) \rightarrow \mathrm{ad}\, x \in A\underline{\mathrm{u}}x < \infty) \wedge$

.1 $(\bigwedge x \in A(\underline{\mathrm{u}}x > -\infty) \rightarrow \mathrm{ad}\, x \in A\underline{\mathrm{u}}x > -\infty) \wedge$

.2 $(\bigwedge x \in A(\underline{\mathrm{u}}x \geq 0) \rightarrow \mathrm{ad}\, x \in A\underline{\mathrm{u}}x \geq 0) \wedge$

.3 $(\bigwedge x \in A(\underline{\mathrm{u}}x = 0) \rightarrow \mathrm{ad}\, x \in A\underline{\mathrm{u}}x = 0) \wedge$

.4 $(\bigwedge x \in A(0 \leq \underline{\mathrm{u}}x \leq \underline{\mathrm{v}}x) \rightarrow \mathrm{ad}\, x \in A\underline{\mathrm{u}}x \leq \mathrm{ad}\, x \in A\underline{\mathrm{v}}x) \wedge$

.5 $(\mathrm{ad}\, x \in A\underline{\mathrm{u}}x \in \mathrm{kt} \wedge B \subset A \rightarrow \mathrm{ad}\, x \in B\underline{\mathrm{u}}x \in \mathrm{kt}) \wedge$

.6 $(\mathrm{ad}\, x \in A\underline{\mathrm{u}}x \in \mathrm{kt} \rightarrow \bigwedge x \in A(\underline{\mathrm{u}}x \in \mathrm{kt})) \wedge$

.7 $|\,\mathrm{ad}\, x \in A\underline{\mathrm{u}}x| \leq \mathrm{ad}\, x \in A|\underline{\mathrm{u}}x| \wedge$

.8 $(\bigwedge x \in A(\underline{\mathrm{u}}x \geq 0) \wedge B \subset A \rightarrow \mathrm{ad}\, x \in B\underline{\mathrm{u}}x \leq \mathrm{ad}\, x \in A\underline{\mathrm{u}}x))$

Infinite Summation

1.65 Definitions

.0 $(\mathrm{ps}\, x \equiv \mathrm{Inf}\, \mathsf{E}\, t(0 \le t \ge x))$

.1 $(\mathrm{ng}\, x \equiv \mathrm{ps} - x)$

1.66 Definitions

.0 $(\mathrm{ril}\, y \equiv (y \in \mathrm{rl} \vee y = \mathrm{U}))$

.1 $(\mathrm{rilp}\, y \equiv (y \ge 0 \vee y = \mathrm{U})$

1.67 Definitions

.0 $(\mathrm{sumrun} \equiv \mathsf{E}\, \alpha, \beta(\alpha \subset \beta \in \mathrm{fnt}))$

Note that

direction is sumrun

and hence

run is sumrun .

.1 $(\mathrm{Ad}\, x\underline{\mathrm{u}}x \equiv (\mathrm{lm}\, \alpha\, \mathrm{sumrun}\, \mathrm{ad}\, x \in \alpha\, \mathrm{ps}\, \underline{\mathrm{u}}x — \ \mathrm{lm}\, \alpha\, \mathrm{sumrun}\, \mathrm{ad}\, x \in \alpha\, \mathrm{ng}\, \underline{\mathrm{u}}x))$

.2 $(\sum x\underline{\mathrm{u}}x \equiv (\mathrm{Ad}\, x\, \mathrm{prt}''\, \underline{\mathrm{u}}x \in \mathrm{rf} \to \mathrm{Ad}\, x\, \mathrm{prt}'\, \underline{\mathrm{u}}x + \mathrm{i} \cdot \mathrm{Ad}\, x\, \mathrm{prt}''\, \underline{\mathrm{u}}x))$

1.68 Theorems

.0 $(0 \le x \to \mathrm{ps}\, x = x)$

.1 $(x \le 0 \to \mathrm{ps}\, x = 0)$

.2 $(x \backsim\in \mathrm{rl} \to \mathrm{ps}\, x = \infty)$

.3 $(0 \le \mathrm{ps}\, x)$

.4 $(0 \le \mathrm{ng}\, x)$

.5 $(\mathrm{ril}\, x \to x = \mathrm{ps}\, x — \ \mathrm{ng}\, x)$

.6 $(\mathrm{ril}\, x \to |x| = \mathrm{ps}\, x + \mathrm{ng}\, x)$

.7 $(\mathrm{ps}(x + y) + \mathrm{ng}\, x + \mathrm{ng}\, y = \mathrm{ng}(x + y) + \mathrm{ps}\, x + \mathrm{ps}\, y)$

.8 $(\mathrm{ng} - x = \mathrm{ps}\, x)$

.9 $(0 \le \mathrm{ps}(x + y) \le \mathrm{ps}\, x + \mathrm{ps}\, y)$

.10 $(0 < c < \infty \to \mathrm{ps}(c \cdot x) = c \cdot \mathrm{ps}\, x \wedge \mathrm{ng}(c \cdot x) = c \cdot \mathrm{ng}\, x)$

.11 $(0 \le \mathrm{ng}(x + y) \le \mathrm{ng}\, x + \mathrm{ng}\, y)$

Theorem

1.69 $(\bigwedge x \in \mathrm{U}(\underline{\mathrm{u}}x = \underline{\mathrm{v}}x) \to \mathrm{Ad}\, x\underline{\mathrm{u}}x = \mathrm{Ad}\, x\underline{\mathrm{v}}x \wedge \sum x\underline{\mathrm{u}}x = \sum x\underline{\mathrm{v}}x)$

Theorem

1.70 $(\bigwedge x \in \mathrm{U}(0 \le \underline{\mathrm{u}}x) \to 0 \le \mathrm{Ad}\, x\underline{\mathrm{u}}x = \sup \alpha \in \mathrm{fnt}\, \mathrm{ad}\, x \in \alpha\underline{\mathrm{u}}x)$

Theorem

1.71 $(\mathrm{Ad}\, x\underline{\mathrm{u}}x = \mathrm{Ad}\, x\, \mathrm{ps}\, \underline{\mathrm{u}}x — \ \mathrm{Ad}\, x\, \mathrm{ng}\, \underline{\mathrm{u}}x)$

1.72 Theorems

.0 $(\bigvee x \in \mathrm{U}(\underline{\mathrm{u}}x \backsim\in \mathrm{rl}) \to \mathrm{Ad}\, x\, \mathrm{ps}\, \underline{\mathrm{u}}x = \mathrm{Ad}\, x\, \mathrm{ng}\, \underline{\mathrm{u}}x = \infty)$

.1 $(\bigvee x \in \mathrm{U}(\underline{\mathrm{u}}x \backsim\in \mathrm{rl}) \to \mathrm{Ad}\, x\underline{\mathrm{u}}x = \mathrm{U})$

.2 $(\bigvee x \in \mathrm{U}(\underline{\mathrm{u}}x \backsim\in \mathrm{spl}) \to \sum x\underline{\mathrm{u}}x = \mathrm{U})$

.3 $(\bigwedge x \in \mathrm{U}\, \mathrm{ril}\, \underline{\mathrm{u}}x \to \sum x\underline{\mathrm{u}}x = \mathrm{Ad}\, x\underline{\mathrm{u}}x)$

.4 $(\bigwedge x \in \mathrm{U}\, \mathrm{ril}\, \underline{\mathrm{u}}x \to \mathrm{ril} \sum x\underline{\mathrm{u}}x \wedge \sum x\underline{\mathrm{u}}x = \sum x\, \mathrm{ps}\, \underline{\mathrm{u}}x — \ \sum x\, \mathrm{ng}\, \underline{\mathrm{u}}x)$

.5 $(\mathrm{U} \ne \sum x\, \mathrm{ps}\, \underline{\mathrm{u}}x — \ \sum x\, \mathrm{ng}\, \underline{\mathrm{u}}x \to \sum x\, \mathrm{ps}\, \underline{\mathrm{u}}x — \ \sum x\, \mathrm{ng}\, \underline{\mathrm{u}}x = \sum x\underline{\mathrm{u}}x)$

.6 $(\bigwedge x \in \mathrm{U}\, \mathrm{ril}\, \underline{\mathrm{u}}x \wedge \mathrm{U} \neq \sum x\underline{\mathrm{u}}x \rightarrow \bigwedge x \in \mathrm{U}(\underline{\mathrm{u}}x \in \mathrm{rl}))$
.7 $(\bigwedge x \in \mathrm{U}\, \mathrm{ril}\, \underline{\mathrm{u}}x \wedge \sum x\underline{\mathrm{u}}x \in \mathrm{rf} \rightarrow \bigwedge x \in \mathrm{U}(\underline{\mathrm{u}}x \in \mathrm{rf}))$
.8 $(\sum x\, \mathrm{prt}''\, \underline{\mathrm{u}}x \in \mathrm{rf} \rightarrow \sum x\underline{\mathrm{u}}x = \sum x\, \mathrm{prt}'\, \underline{\mathrm{u}}x + \mathrm{i} \cdot \sum x\, \mathrm{prt}''\, \underline{\mathrm{u}}x)$
.9 $(\sum x\, \mathrm{prt}''\, \underline{\mathrm{u}}x \backsim\in \mathrm{rf} \rightarrow \sum x\underline{\mathrm{u}}x = \mathrm{U})$
.10 $(\sum x\underline{\mathrm{u}}x \neq \mathrm{U} \rightarrow \bigwedge x \in \mathrm{U}(\underline{\mathrm{u}}x \in \mathrm{spl}) \wedge \sum x\underline{\mathrm{u}}x \in \mathrm{spl})$
.11 $(\sum x\underline{\mathrm{u}}x \in \mathrm{kf} \rightarrow \bigwedge x \in \mathrm{U}(\underline{\mathrm{u}}x \in \mathrm{kf}))$
.12 $(\bigwedge x \in \mathrm{U}(0 \leq \underline{\mathrm{u}}x) \rightarrow\ 0 \leq \sum x\underline{\mathrm{u}}x = \sup \alpha \in \mathrm{fnt}\, \mathrm{ad}\, x \in \alpha\underline{\mathrm{u}}x = \mathrm{lm}\, \alpha\, \mathrm{sumrun}\, \mathrm{ad}\, x \in \alpha\underline{\mathrm{u}}x)$
.13 $(\mathrm{Ad}\, x - \underline{\mathrm{u}}x = -\, \mathrm{Ad}\, x\underline{\mathrm{u}}x)$
.14 $(c \in \mathrm{rf} \sim 1 \rightarrow \mathrm{Ad}\, x(c \cdot \underline{\mathrm{u}}x) = c \cdot \mathrm{Ad}\, x\underline{\mathrm{u}}x)$

Theorem
1.73 $(\sum x\underline{\mathrm{u}}x \in \mathrm{spl} \rightarrow \sum x\underline{\mathrm{u}}x = \mathrm{lm}\, \alpha\, \mathrm{sumrun}\, \mathrm{ad}\, x \in \alpha\underline{\mathrm{u}}x)$

Theorem
1.74 $(c \in \mathrm{rf} \sim 1 \rightarrow \sum x(c \cdot \underline{\mathrm{u}}x) = c \cdot \sum x\underline{\mathrm{u}}x)$

Theorem
1.75 $(\sum x - \underline{\mathrm{u}}x = -\sum x\underline{\mathrm{u}}x)$

Theorem
1.76 $(\sum x0 = 0)$

Thoerem
1.77 $(\bigwedge x \in \mathrm{U}(0 \leq \underline{\mathrm{u}}x \leq \underline{\mathrm{v}}x) \rightarrow \sum x(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) = \sum x\underline{\mathrm{u}}x + \sum x\underline{\mathrm{v}}x)$

Theorem
1.78 $(\bigwedge x \in \mathrm{U}(0 \leq \underline{\mathrm{u}}x \leq \underline{\mathrm{v}}x) \rightarrow 0 \leq \sum x\underline{\mathrm{u}}x \leq \sum x\underline{\mathrm{v}}x)$

Lemma
1.79 $(\bigwedge x \in \mathrm{U}(\mathrm{ril}\, \underline{\mathrm{u}}x \wedge \mathrm{ril}\, \underline{\mathrm{v}}x) \wedge -\infty < \sum x\underline{\mathrm{u}}x + \sum x\underline{\mathrm{v}}x = s \rightarrow s = \sum x(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x))$

Theorem
1.80 $(\bigwedge x \in \mathrm{U}(\mathrm{ril}\, \underline{\mathrm{u}}x \wedge \mathrm{ril}\, \underline{\mathrm{v}}x) \wedge \sum x\underline{\mathrm{u}}x + \sum x\underline{\mathrm{v}}x \in \mathrm{rl} \rightarrow \sum x(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) = \sum x\underline{\mathrm{u}}x + \sum x\underline{\mathrm{v}}x)$

1.81 Theorems
.0 $(\mathrm{ril}\, a \wedge \mathrm{ril}\, b \rightarrow a + \mathrm{i} \cdot b \in \mathrm{spl} \leftrightarrow a \in \mathrm{rl} \wedge b \in \mathrm{rf})$
.1 $(x + y \in \mathrm{spl} \rightarrow |\, \mathrm{prt}''(x + y)| \leq |\, \mathrm{prt}''\, x| + |\, \mathrm{prt}''\, y|)$
.2 $(\sum x\underline{\mathrm{u}}x \in \mathrm{spl} \rightarrow \mathrm{prt}' \sum x\underline{\mathrm{u}}x = \mathrm{prt}' \sum x\underline{\mathrm{u}}x)$
.3 $(\sum x\underline{\mathrm{u}}x \in \mathrm{kf} \rightarrow \mathrm{prt}'' \sum x\underline{\mathrm{u}}x = \sum \mathrm{prt}''\, \underline{\mathrm{u}}x)$
.4 $(\sum x\underline{\mathrm{u}}x = \sum x(\mathrm{prt}'\, \underline{\mathrm{u}}x + \mathrm{i} \cdot \mathrm{prt}''\, \underline{\mathrm{u}}x))$
Hint: Use 1.72.10, 1.15.4, .0, and 1.15.0.

1.82 Theorems
.0 $(\bigwedge x \in \mathrm{U}\, \mathrm{ril}\, \underline{\mathrm{u}}x \rightarrow \sum x\underline{\mathrm{u}}x \in \mathrm{rf} \leftrightarrow \sum x|\underline{\mathrm{u}}x| < \infty)$
.1 $(\sum x\underline{\mathrm{u}}x \in \mathrm{kf} \leftrightarrow \sum x|\underline{\mathrm{u}}x| < \infty)$

Theorem
1.83 $(|\sum x\underline{\mathrm{u}}x| \leq \sum x|\underline{\mathrm{u}}x|)$

Theorem
1.84 $(c \in \mathrm{kf} \wedge \sum x\underline{\mathrm{u}}x \in \mathrm{kf} \rightarrow \sum x(c \cdot \underline{\mathrm{u}}x) = c \cdot \sum x\underline{\mathrm{u}}x)$

Theorem
1.85 $(\sum x\underline{\mathrm{u}}x + \sum x\underline{\mathrm{v}}x \in \mathrm{spl} \rightarrow \sum x(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) = \sum x\underline{\mathrm{u}}x + \sum x\underline{\mathrm{v}}x)$

Proof:

Let

$$(a = \mathrm{prt}'(\sum x\underline{\mathrm{u}}x + \sum x\underline{\mathrm{v}}x) \wedge b = \mathrm{prt}''(\sum x\underline{\mathrm{u}}x + \sum x\underline{\mathrm{v}}x) \wedge \alpha = \sum x\,\mathrm{prt}'(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) \wedge \beta = \sum x\,\mathrm{prt}''(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x))\ .$$

We know

.0 $(a \in \mathrm{rl} \wedge b \in \mathrm{rf} \wedge$

.1 $a + \mathrm{i} \cdot b = \sum x\underline{\mathrm{u}}x + \sum x\underline{\mathrm{v}}x \wedge$

$$\begin{aligned} \mathrm{rl} \ni a &= \mathrm{prt}' \sum x\underline{\mathrm{u}}x + \mathrm{prt}' \sum x\underline{\mathrm{v}}x \\ &= \sum x\,\mathrm{prt}'\,\underline{\mathrm{u}}x + \sum x\,\mathrm{prt}'\,\underline{\mathrm{v}}x \\ &= \sum x(\mathrm{prt}'\,\underline{\mathrm{u}}x + \mathrm{prt}'\,\underline{\mathrm{v}}x) \\ &= \sum x\,\mathrm{prt}'(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) \\ &= \alpha) \end{aligned}$$

Accordingly

.2 $(\alpha = a \wedge$
$(\bigwedge x \in \mathrm{U}(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x \in \mathrm{spl}))$.

Thus

$$(\sum x|\,\mathrm{prt}''(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x)| \leq \sum x(|\,\mathrm{prt}''\,\underline{\mathrm{u}}x| + |\,\mathrm{prt}''\,\underline{\mathrm{v}}x|) = \sum x|\,\mathrm{prt}''\,\underline{\mathrm{u}}x| + \sum x|\,\mathrm{prt}''\,\underline{\mathrm{v}}x| < \infty)\ .$$

Consequently

.3 $(\beta \in \mathrm{rf})$

and, because of .3 and .2,

.4 $(\sum x(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) = a + \mathrm{i} \cdot b)$.

Now

$$\begin{aligned} (a \in \mathrm{rf} & \\ &\rightarrow \sum x\underline{\mathrm{u}}x + \sum x\underline{\mathrm{v}}x \in \mathrm{kf} \\ &\rightarrow b = \sum x\,\mathrm{prt}''\,\underline{\mathrm{u}}x + \sum x\,\mathrm{prt}''\,\underline{\mathrm{v}}x \\ &\qquad = \sum x(\mathrm{prt}''\,\underline{\mathrm{u}}x + \mathrm{prt}''\,\underline{\mathrm{v}}x) \\ &\qquad = \sum x\,\mathrm{prt}''(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) \\ &\qquad = \beta) \end{aligned}$$

and hence from this, .4, and .1, it follows that

.5 $(a \in \mathrm{rf} \rightarrow \sum x(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) = \sum x\underline{\mathrm{u}}x + \sum x\underline{\mathrm{v}}x)$.

From .3, .0, and .1 we deduce

.6 $(a \in \mathrm{infin} \rightarrow$

$$\begin{aligned} &\sum x(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) \\ &\quad = a + \mathrm{i} \cdot \beta \\ &\quad = a \\ &\quad = a + \mathrm{i} \cdot b \\ &\quad = \sum x\underline{\mathrm{u}}x + \sum x\underline{\mathrm{v}}x) \end{aligned}$$

Because of .5, .6, and .0, the desired conclusion is at hand.

Thoerem
1.86 $(\bigwedge x \in \mathrm{U}\,\mathrm{rilp}\,\underline{\mathrm{u}}x \rightarrow \sum x\underline{\mathrm{u}}x = 0 \leftrightarrow \bigwedge x \in \mathrm{U}(\underline{\mathrm{u}}x = 0))$

Definition
1.87 $((a \bullet b) \equiv (a \neq 0 \wedge b \neq 0 \wedge a \cdot b))$

1.88 Theorems
.0 $(a \neq 0 \wedge b \neq 0 \rightarrow a \bullet b = a \cdot b)$
.1 $(0 < |a| < \infty \rightarrow a \bullet b = a \cdot b)$
.2 $(a \bullet b = b \bullet a)$
.3 $(a \bullet 0 = 0 \bullet a = 0)$
.4 $(a \bullet (b \bullet c) = (a \bullet b) \bullet c)$
.5 $(c \in \mathrm{kf} \rightarrow c \bullet (a + b) = c \bullet a + c \bullet b)$
.6 $(\mathrm{rilp}\, a \wedge \mathrm{rilp}\, b \wedge \mathrm{rilp}\, c \rightarrow c \bullet (a + b) = c \bullet a + c \bullet b)$
.7 $(\mathrm{rilp}\, c \rightarrow \mathrm{ps}(c \bullet x) = c \bullet \mathrm{ps}\, x \wedge \mathrm{ng}(c \bullet x) = c \bullet \mathrm{ng}\, x)$
.8 $(a \leq b \wedge 0 \leq c \rightarrow c \bullet a \leq c \bullet b)$
.9 $(a \in \mathrm{rl} \wedge b \in \mathrm{rl} \rightarrow a \bullet b \in \mathrm{rl})$
.10 $(\mathrm{ril}\, a \wedge \mathrm{ril}\, b \rightarrow \mathrm{ril}(a \bullet b))$
.11 $(a \bullet c + b \bullet c \in \mathrm{kt} \rightarrow a \bullet c + b \bullet c = (a + b) \bullet c)$
Hint. Use 1.11.82.
.12 $(|x \bullet y| = |x| \bullet |y|)$
.13 $(\mathrm{ril}\, r \wedge r \bullet c \in \mathrm{spl} \rightarrow r = 0 \vee c \in \mathrm{spl})$
Hint. Use 1.11.83.

1.89 Theorems
.0 $(r \in \mathrm{rf} \rightarrow \mathrm{prt}'(c \bullet r) = \mathrm{prt}'\, c \bullet r \wedge \mathrm{prt}''(c \bullet r) = \mathrm{prt}''\, c \bullet r)$
.1 $(r \in \mathrm{rl} \wedge \mathrm{prt}''\, c \bullet r \in \mathrm{rf} \rightarrow c \bullet r \in \mathrm{spl})$
.2 $(\mathrm{ril}\, r \wedge c \bullet r \in \mathrm{spl} \rightarrow \mathrm{prt}'(c \bullet r) = \mathrm{prt}'\, c \bullet r \wedge \mathrm{prt}''(c \bullet r) = \mathrm{prt}''\, c \bullet r)$

1.90 Theorems
.0 $(c \in \mathrm{rf} \rightarrow \sum(c \bullet \underline{\mathrm{u}}x) = c \bullet \sum x\underline{\mathrm{u}}x)$
.1 $(\mathrm{ril}\, c \wedge \bigwedge c \in \mathrm{U}\, \mathrm{rilp}\, \underline{\mathrm{u}}x \rightarrow \sum x(c \bullet \underline{\mathrm{u}}x) = c \bullet \sum x\underline{\mathrm{u}}x)$
.2 $(|c| \bullet \sum x|\underline{\mathrm{u}}x| < \infty \rightarrow \sum x(c \bullet \underline{\mathrm{u}}x) = c \bullet \sum x\underline{\mathrm{u}}x)$
.3 $(\sum x(c \bullet \underline{\mathrm{u}}x) \in \mathrm{spl} \wedge c \bullet \sum x\underline{\mathrm{u}}x \in \mathrm{spl} \rightarrow \sum(c \bullet \underline{\mathrm{u}}x) = c \bullet \sum x\underline{\mathrm{u}}x)$

1.91 Theorems
.0 $(\bigwedge x \in \mathrm{U}\, \mathrm{rilp}\, \underline{\mathrm{u}}x \wedge c \bullet \sum x\underline{\mathrm{u}}x \in \mathrm{spl} \rightarrow \sum x(c \bullet \underline{\mathrm{u}}x) = c \bullet \sum x\underline{\mathrm{u}}x)$
.1 $(\bigwedge x \in \mathrm{U}\, \mathrm{rilp}\, \underline{\mathrm{u}}x \wedge \sum x(c \bullet \underline{\mathrm{u}}x) \in \mathrm{spl} \rightarrow \sum x(c \bullet \underline{\mathrm{u}}x) = c \bullet \sum x\underline{\mathrm{u}}x)$
Proof:
Because of 1.89.1 and .0
$$(\mathrm{rf} \ni \sum x\, \mathrm{prt}''(c \bullet \underline{\mathrm{u}}x) = \sum x(\mathrm{prt}''\, c \bullet \underline{\mathrm{u}}x) = \mathrm{prt}''\, c \bullet \sum x\underline{\mathrm{u}}x$$
$$\rightarrow c \bullet \sum x\underline{\mathrm{u}}x \in \mathrm{spl}$$
$$\rightarrow c \bullet \sum x\underline{\mathrm{u}}x = \sum x(c \bullet \underline{\mathrm{u}}x))\ .$$

Definition
1.92 $(\mathrm{Cr}\, xy \equiv (x \in y \wedge 1)$

1.93 Theorems
.0 $(\mathrm{Cr}\, xy = 1 \leftrightarrow x \in y)$
.1 $(\mathrm{Cr}\, xy = 0 \leftrightarrow x \backsim\!\in y)$

Definition
1.94 $(\sum x \in A\underline{\mathrm{u}}x \equiv \sum x(\mathrm{Cr}\, xA \bullet \underline{\mathrm{u}}x))$

1.95 Theorems

.0 $(\bigwedge x \in A(\underline{\mathrm{u}}x = \underline{\mathrm{v}}x) \to \sum x \in A\underline{\mathrm{u}}x = \sum x \in A\underline{\mathrm{v}}x)$

.1 $(\sum x \in A\underline{\mathrm{u}}x \neq \mathrm{U} \to \bigwedge x \in A(\underline{\mathrm{u}}x \in \mathrm{spl}) \wedge \sum x \in A\underline{\mathrm{u}}x \in \mathrm{spl})$

.2 $(\sum x \in A\underline{\mathrm{u}}x \in \mathrm{kf} \to \bigwedge x \in A(\underline{\mathrm{u}}x \in \mathrm{kf}))$

.3 $(\bigwedge x \in A\, \mathrm{ril}\, \underline{\mathrm{u}}x \wedge \mathrm{U} \neq \sum x \in A\underline{\mathrm{u}}x \to \bigwedge x \in A(\underline{\mathrm{u}}x \in \mathrm{rl}))$

.4 $(\bigwedge x \in A\, \mathrm{ril}\, \underline{\mathrm{u}}x \wedge \sum x \in A\underline{\mathrm{u}}x \in \mathrm{rf} \to \bigwedge x \in A(\underline{\mathrm{u}}x \in \mathrm{rf}))$

.5 $(\bigwedge x \in A\, \mathrm{ril}\, \underline{\mathrm{u}}x \to \mathrm{ril} \sum x \in A\underline{\mathrm{u}}x \wedge \sum x \in A\underline{\mathrm{u}}x = \sum x \in A\, \mathrm{ps}\, \underline{\mathrm{u}}x \;—\; \sum x \in A\, \mathrm{ng}\, \underline{\mathrm{u}}x)$

.6 $(\sum x \in A\, \mathrm{prt}''\, \underline{\mathrm{u}}x \in \mathrm{rf} \to \sum x \in A\underline{\mathrm{u}}x = \sum x \in A\, \mathrm{prt}'\, \underline{\mathrm{u}}x + \mathrm{i} \cdot \sum x \in A\, \mathrm{prt}''\, \underline{\mathrm{u}}x)$

.7 $(\sum x \in A\, \mathrm{prt}''\, \underline{\mathrm{u}}x \backsim\in \mathrm{rf} \to \sum x \in A\underline{\mathrm{u}}x = \mathrm{U})$

.8 $(\sum x \in A\underline{\mathrm{u}}x \in \mathrm{spl} \to \mathrm{prt}' \sum x \in A\underline{\mathrm{u}}x = \sum x \in A\, \mathrm{prt}'\, \underline{\mathrm{u}}x)$

.9 $(\sum x \in A\underline{\mathrm{u}}x \in \mathrm{kf} \to \mathrm{prt}'' \sum x \in A\underline{\mathrm{u}}x = \sum x \in A\, \mathrm{prt}''\, \underline{\mathrm{u}}x)$

.10 $(\bigwedge x \in A\, \mathrm{rilp}\, \underline{\mathrm{u}}x \to \sum x \in A\underline{\mathrm{u}}x = 0 \leftrightarrow \bigwedge x \in A(\underline{\mathrm{u}}x = 0))$

.11 $(\bigwedge x \in A\, \mathrm{ril}\, \underline{\mathrm{u}}x \to \sum x \in A\underline{\mathrm{u}}x \in \mathrm{rf} \leftrightarrow \sum x \in A|\underline{\mathrm{u}}x| < \infty)$

.12 $(\sum x \in A\underline{\mathrm{u}}x \in \mathrm{kf} \leftrightarrow \sum x \in A|\underline{\mathrm{u}}x| < \infty)$

.13 $(|\sum x \in A\underline{\mathrm{u}}x| \leq \sum x \in A|\underline{\mathrm{u}}x|)$

.14 $(c \in \mathrm{rf} \to \sum x \in A(c \bullet \underline{\mathrm{u}}x) = c \bullet \sum x \in A\underline{\mathrm{u}}x)$

.15 $(0 \neq c \in \mathrm{rf} \to \sum x \in A(c \cdot \underline{\mathrm{u}}x) = c \cdot \sum x \in A\underline{\mathrm{u}}x)$

.16 $(c \in \mathrm{kf} \wedge \sum x \in A\underline{\mathrm{u}}x \in \mathrm{kf} \to \sum x \in A(c \cdot \underline{\mathrm{u}}x) = c \cdot \sum x \in A\underline{\mathrm{u}}x)$

.17 $(\mathrm{ril}\, c \wedge \bigwedge x \in A\, \mathrm{rilp}\, \underline{\mathrm{u}}x \to \sum x \in A(c \bullet \underline{\mathrm{u}}x) = c \bullet \sum x \in A\underline{\mathrm{u}}x)$

.18 $(\bigwedge x \in A\, \mathrm{rilp}\, \underline{\mathrm{u}}x \wedge c \bullet \sum x \in A\underline{\mathrm{u}}x \in \mathrm{spl} \to \sum x \in A(c \bullet \underline{\mathrm{u}}x) = c \bullet \sum x \in A\underline{\mathrm{u}}x)$

.19 $(\bigwedge x \in A\, \mathrm{rilp}\, \underline{\mathrm{u}}x \wedge \sum x \in A(c \bullet \underline{\mathrm{u}}x) \in \mathrm{spl} \to \sum x \in A(c \bullet \underline{\mathrm{u}}x) = c \bullet \sum x \in A\underline{\mathrm{u}}x)$

.20 $(|c| \bullet \sum x \in A|\underline{\mathrm{u}}x| < \infty \to \sum x \in A(c \bullet \underline{\mathrm{u}}x) = c \bullet \sum x \in A\underline{\mathrm{u}}x)$

.21 $(\sum x \in A(c \bullet \underline{\mathrm{u}}x) \in \mathrm{spl} \wedge c \bullet \sum x \in A\underline{\mathrm{u}}x \in \mathrm{spl} \to \sum x \in A(c \bullet \underline{\mathrm{u}}x) = c \bullet \sum x \in A\underline{\mathrm{u}}x)$

.22 $(\sum x\underline{\mathrm{u}}x = \sum x \in \mathrm{U}\underline{\mathrm{u}}x)$

.23 $(\sum x \in 0\underline{\mathrm{u}}x = 0)$

.24 $(s = \sum x \in A\underline{\mathrm{u}}x + \sum x \in A\underline{\mathrm{v}}x \in \mathrm{spl} \to \sum x \in A(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) = s)$

Hint. Use 1.88.5

.25 $(x = \sum x \in A\underline{\mathrm{v}}x + \sum x \in A(\underline{\mathrm{u}}x \;—\; \underline{\mathrm{v}}x) \in \mathrm{spl} \to \sum x \in A\underline{\mathrm{u}}x = s)$

.26 $(s = \sum x \in A(\underline{\mathrm{u}}x \bullet \underline{\mathrm{w}}x) + \sum x \in A(\underline{\mathrm{v}}x \bullet \underline{\mathrm{w}}x) \in \mathrm{spl} \to \sum x \in A((\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) \bullet \underline{\mathrm{w}}x) = s)$

.27 $(s = \sum x \in A(\underline{\mathrm{v}}x \bullet \underline{\mathrm{w}}x) + \sum x \in A((\underline{\mathrm{u}}x \;—\; \underline{\mathrm{v}}x) \bullet \underline{\mathrm{w}}x) \in \mathrm{spl} \to \sum x \in A(\underline{\mathrm{u}}x \bullet \underline{\mathrm{w}}x) = s)$

.28 $(\bigwedge x \in A(\underline{\mathrm{u}}x \geq 0) \to \sum x \in A\underline{\mathrm{u}}x \geq 0)$

.29 $(\bigwedge x \in A\, \mathrm{rilp}\, \underline{\mathrm{u}}x \to \mathrm{rilp} \sum x \in A\underline{\mathrm{u}}x)$

.30 $(\bigwedge x \in A(\underline{\mathrm{u}}x \leq \underline{\mathrm{v}}x) \to \sum x \in A\underline{\mathrm{v}}x \backsim< \sum x \in A\underline{\mathrm{u}}x)$

.31 $(\bigwedge x \in A(0 \leq \underline{\mathrm{u}}x \leq \underline{\mathrm{v}}x) \to \sum x \in A\underline{\mathrm{u}}x \leq \sum x \in A\underline{\mathrm{v}}x)$

Theorem
1.96 $(\bigwedge x \in B(\underline{\mathrm{u}}x \geq 0) \to 0 \leq \sum x \in B\underline{\mathrm{u}}x = \sup \alpha \in \mathrm{fnt} \cap \mathrm{sb}\, B\, \mathrm{ad}\, x \in \alpha\underline{\mathrm{u}}x)$

Theorem
1.97 $(A \in \mathrm{fnt} \wedge \bigwedge x \in A(\underline{\mathrm{u}}x \geq 0) \to 0 \leq \mathrm{ad}\, x \in A\underline{\mathrm{u}}x = \sum x \in A\underline{\mathrm{u}}x)$

Theorem
1.98 $(\bigwedge x \in B(\underline{\mathrm{u}}x \geq 0) \to \sum x \in B\underline{\mathrm{u}}x = \sup \alpha \in \mathrm{fnt} \cap \mathrm{sb}\, B \sum x \in \alpha\underline{\mathrm{u}}x)$

Lemma
1.99 $(A \in \mathrm{fnt} \wedge \bigwedge x \in A\, \mathrm{ril}\, \underline{\mathrm{u}}x \rightarrow \sum x \in A\underline{\mathrm{u}}x = \mathrm{ad}\, x \in A\underline{\mathrm{u}}x)$

Theorem
1.100 $(A \in \mathrm{fnt} \wedge \bigwedge x \in A(\underline{\mathrm{u}}x \in \mathrm{spl}) \rightarrow \sum x \in A\underline{\mathrm{u}}x = \mathrm{ad}\, x \in A\underline{\mathrm{u}}x)$
Easily seen now is the following
Lemma
1.101 $(A \cap B = 0 \wedge A \cup B \in \mathrm{fnt} \wedge \bigwedge x \in A \cup B(\underline{\mathrm{u}}x \in \mathrm{spl}) \rightarrow \sum x \in A \cup B\underline{\mathrm{u}}x = \sum x \in A\underline{\mathrm{u}}x + \sum x \in B\underline{\mathrm{u}}x)$

Theorem
1.102 $(y \in \mathrm{U} \wedge \underline{\mathrm{u}}y \in \mathrm{spl} \rightarrow \sum x \in \mathrm{sng}\, y\underline{\mathrm{u}}x = \underline{\mathrm{u}}y)$

1.103 Theorems
.0 $(\bigwedge x \in A \bigwedge y \in B(\underline{\mathrm{u}}'xy = \underline{\mathrm{v}}'xy) \rightarrow \sum x \in A \sum y \in B\underline{\mathrm{u}}'xy = \sum x \in A \sum y \in B\underline{\mathrm{v}}'xy)$
.1 $(s = \sum x \in A \sum y \in B\underline{\mathrm{u}}'xy + \sum x \in A \sum y \in B\underline{\mathrm{v}}'xy \in \mathrm{spl} \rightarrow \sum x \in A \sum y \in B(\underline{\mathrm{u}}'xy + \underline{\mathrm{v}}'xy) = s)$
.2 $(s = \sum x \in A \sum y \in B\underline{\mathrm{v}}'xy + \sum x \in A \sum y \in B(\underline{\mathrm{u}}'xy - \underline{\mathrm{v}}'xy) \in \mathrm{spl} \rightarrow \sum x \in A \sum y \in B\underline{\mathrm{u}}'xy = s)$
.3 $(|\sum x \in A \sum y \in B\underline{\mathrm{u}}'xy| \leq \sum x \in A \sum y \in B|\underline{\mathrm{u}}'xy|)$
.4 $(s = \sum x \in A \sum y \in B(\underline{\mathrm{u}}'xy \bullet \underline{\mathrm{w}}'xy) + \sum x \in A \sum y \in B(\underline{\mathrm{v}}'xy \bullet \underline{\mathrm{w}}'xy) \in \mathrm{spl}$
$\rightarrow \sum x \in A \sum y \in B((\underline{\mathrm{u}}'xy + \underline{\mathrm{v}}'xy) \bullet \underline{\mathrm{w}}'xy) = s)$
.5 $(s = \sum x \in A \sum y \in B(\underline{\mathrm{v}}'xy \bullet \underline{\mathrm{w}}'xy) + \sum x \in A \sum y \in B((\underline{\mathrm{u}}'xy - \underline{\mathrm{v}}'xy) \bullet \underline{\mathrm{w}}'xy)$
$\rightarrow \sum x \in A \sum y \in B(\underline{\mathrm{u}}'xy \bullet \underline{\mathrm{w}}'xy) = s)$
.6 $(\sum x \sum y\underline{\mathrm{u}}'xy = \sum x \in \mathrm{U} \sum y \in \mathrm{U}\underline{\mathrm{u}}'xy)$

Lemma
1.104 $(\bigwedge x \bigwedge y(\underline{\mathrm{u}}'xy \geq 0) \wedge A \in \mathrm{fnt} \rightarrow \sum x \in A \sum y\underline{\mathrm{u}}'xy = \sum y \sum x \in A\underline{\mathrm{u}}'xy)$
Hint. Use 1.101 and 1.77 and induction in $\mathrm{pwr}'\, A$.

Lemma
1.105 $(\bigwedge x \bigwedge y(\underline{\mathrm{u}}'xy \geq 0) \rightarrow \sum x \sum y\underline{\mathrm{u}}'xy \leq \sum y \sum x\underline{\mathrm{u}}'xy)$

Theorem
1.106 $(\bigwedge x \bigwedge y(\underline{\mathrm{u}}'xy \geq 0) \rightarrow \sum x \sum y\underline{\mathrm{u}}'xy = \sum y \sum x\underline{\mathrm{u}}'xy)$

1.107 Lemmas
.0 $(\bigwedge x \bigwedge y(\underline{\mathrm{v}}'xy \geq 0) \wedge \bigwedge x(\underline{\mathrm{u}}x \geq 0) \rightarrow \sum x \sum y(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy) = \sum y \sum x(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy))$
.1 $(\bigwedge x \bigwedge y(\underline{\mathrm{v}}'xy \geq 0) \wedge \bigwedge x\, \mathrm{ril}\, \underline{\mathrm{u}}x \wedge s = \sum x \sum y(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy) \in \mathrm{rl}$
$\rightarrow s = \sum y \sum x\, \mathrm{ps}(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy) - \sum y \sum x\, \mathrm{ng}(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy) = \sum y \sum x(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy))$

Lemma
1.108 $(\bigwedge x \bigwedge y(\underline{\mathrm{v}}'xy \geq 0) \wedge s = \sum x \sum y(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy) \in \mathrm{spl}$
$\rightarrow s = \sum y \sum x\, \mathrm{prt}'(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy) + \mathrm{i} \cdot \sum y \sum x\, \mathrm{prt}''(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy) = \sum y \sum x(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy))$
Proof:

Using 1.91, 1.89.2, 1.81.0, 1.107.1, 1.84, 1.103, and 1.81.4

$(\text{spl} \ni s$

$= \sum x \sum y(\underline{\text{u}}x \bullet \underline{\text{v}}'xy)$

$= \sum x(\underline{\text{u}}x \bullet \sum y\underline{\text{v}}'xy)$

$= \sum x\,\text{prt}'(\underline{\text{u}}x \bullet \sum y\underline{\text{v}}'xy) + \text{i} \cdot \sum x\,\text{prt}''(\underline{\text{u}}x \bullet \sum y\underline{\text{v}}'xy)$

$= \sum x(\text{prt}'\,\underline{\text{u}}x \bullet \sum y\underline{\text{v}}'xy) + \text{i} \cdot \sum x(\text{prt}''\,\underline{\text{u}}x \bullet \sum y\underline{\text{v}}'xy)$

$= \sum x \sum y(\text{prt}'\,\underline{\text{u}}x \bullet \underline{\text{v}}'xy) + \text{i} \cdot \sum x \sum y(\text{prt}''\,\underline{\text{u}}x \bullet \underline{\text{v}}'xy)$

$= \sum x \sum y\,\text{prt}'(\underline{\text{u}}x \bullet \underline{\text{v}}'xy) + \text{i} \cdot \sum x \sum y\,\text{prt}''(\underline{\text{u}}x \bullet \underline{\text{v}}'xy)$

$= \sum y \sum x\,\text{prt}'(\underline{\text{u}}x \bullet \underline{\text{v}}'xy) + \text{i} \cdot \sum y \sum x\,\text{prt}''(\underline{\text{u}}x \bullet \underline{\text{v}}'xy)$

$= \sum y \sum x\,\text{prt}'(\underline{\text{u}}x \bullet \underline{\text{v}}'xy) + \sum y \sum x(\text{i} \cdot \text{prt}''(\underline{\text{u}}x \bullet \underline{\text{v}}'xy))$

$= \sum y \sum x(\text{prt}'(\underline{\text{u}}x \bullet \underline{\text{v}}'xy) + (\text{i} \cdot \text{prt}''(\underline{\text{u}}x \bullet \underline{\text{v}}'xy))$

$= \sum y \sum x(\underline{\text{u}}x \bullet \underline{\text{v}}'xy))$

Very useful is

Theorem

1.109 $(\bigwedge x \bigwedge y(\underline{\text{v}}'xy \geq 0) \wedge s = \sum x(\underline{\text{u}}x \bullet \sum y\underline{\text{v}}'xy) \in \text{spl} \rightarrow \sum x \sum y(\underline{\text{u}}x \bullet \underline{\text{v}}'xy) = \sum y \sum x(\underline{\text{u}}x \bullet \underline{\text{v}}'xy))$

Proof:

Using 1.91 and 1/108 we infer

$(\text{spl} \ni s$

$= \sum x(\underline{\text{u}}x \bullet \sum y\underline{\text{v}}'xy)$

$= \sum x \sum y(\underline{\text{u}}x \bullet \underline{\text{v}}'xy)$

$= \sum y \sum x(\underline{\text{u}}x \bullet \underline{\text{v}}'xy))$

A generalization of 1.109 is

Theorem

1.110 $(\bigwedge x \bigwedge y\,\text{rilp}\,\underline{\text{w}}'xy \wedge s = \sum x(\underline{\text{u}}x \bullet \sum y\underline{\text{w}}'xy) \in \text{spl} \rightarrow \sum x \sum y(\underline{\text{u}}x \bullet \underline{\text{w}}'xy) = s = \sum y \sum x(\underline{\text{u}}x \bullet \underline{\text{w}}'xy))$

Proof:

Assume

$(\bigwedge x \bigwedge y(\underline{\text{v}}'xy = |\underline{\text{w}}'xy|)$

and check, with the help of 1.109,

$(\bigwedge x \bigwedge y(\underline{\text{v}}'xy \geq 0) \wedge$

$\bigwedge x \in \text{U}(\underline{\text{u}}x \bullet \sum y\underline{\text{v}}'xy = \underline{\text{u}}x \bullet \sum y\underline{\text{w}}'xy) \wedge$

$\bigwedge x \in \text{U} \bigwedge y \in \text{U}(\underline{\text{u}}x \bullet \underline{\text{v}}'xy = \underline{\text{u}}x \bullet \underline{\text{w}}'xy) \wedge$

$\text{spl} \ni s$

$= \sum x(\underline{\text{u}}x \bullet \sum y\underline{\text{w}}'xy)$

$= \sum x \sum y(\underline{\text{u}}x \bullet \underline{\text{w}}'xy)$

$= \sum x \sum y(\underline{\text{u}}x \bullet \underline{\text{v}}'xy)$

$= s$

$= \sum y \sum x(\underline{\text{u}}x \bullet \underline{\text{v}}'xy)$

$= \sum y \sum x(\underline{\text{u}}x \bullet \underline{\text{w}}'xy))$

We now have almost at once

Summation by Positive Distribution

Theorem

1.111 $(\bigwedge x \in A \bigwedge y \in B\,\text{rilp}\,\underline{\text{v}}'xy \wedge s = \sum x \in A(\underline{\text{u}}x \bullet \sum y \in B\underline{\text{v}}'xy) \in \text{spl}$

$\rightarrow \sum x \in A \sum y \in B(\underline{\text{u}}x \bullet \underline{\text{v}}'xy) = s = \sum y \in B \sum x \in A(\underline{\text{u}}x \bullet \underline{\text{v}}'xy))$

Theorem

1.112 $(\bigwedge x \in A \bigwedge y \in B\,\text{rilp}\,\underline{\text{u}}'xy \rightarrow \sum x \in A \sum y \in B\underline{\text{u}}'xy = \sum y \in B \sum x \in A\underline{\text{u}}'xy)$

Lemma
1.113 $(A \cap B = 0 \wedge s = \sum x \in A\underline{\mathrm{u}}x + \sum x \in B\underline{\mathrm{u}}x \in \mathrm{spl} \to \sum x \in A \cup B\underline{\mathrm{u}}x = s)$
Proof:
Note that
$(\mathrm{Cr}\, x(A \cup B) = \mathrm{Cr}\, xA + \mathrm{Cr}\, xB)$
and use 1.85 and 1.88.11.

An easy consequence of 1.109 is
Theorem
1.114 $(s = \sum x(\underline{\mathrm{u}}x \bullet \sum y \in B\, \mathrm{Cr}\, x\underline{\mathrm{v}}y) \in \mathrm{spl} \to s = \sum y \in B \sum x \in \underline{\mathrm{v}}y\underline{\mathrm{u}}x)$

Summation by Partition
Theorem
1.115 $(S = \bigvee y \in B\underline{\mathrm{v}}y \wedge \bigwedge y \in B \bigwedge n \in B(y \neq n \to \underline{\mathrm{v}}y \cap \underline{\mathrm{v}}n = 0) \wedge \sum x \in S\underline{\mathrm{u}}x \in \mathrm{spl}$
$\to \sum x \in S\underline{\mathrm{u}}x = \sum y \in B \sum x \in \underline{\mathrm{v}}y\underline{\mathrm{u}}x)$
Proof:
Check with the help of 1.95.27 and 1.11.3 that
$(\mathrm{Cr}\, xS = \sum y \in B\, \mathrm{Cr}\, x\underline{\mathrm{v}}y)$
and apply 1.114.

Theorem
1.116 $(S = \bigvee y \in B\underline{\mathrm{v}}y \wedge \bigwedge x \in S(\underline{\mathrm{u}}x \geq 0) \to 0 \leq \sum x \in S\underline{\mathrm{u}}x \leq \sum y \in B \sum x \in \underline{\mathrm{v}}y\underline{\mathrm{u}}x)$
Proof:
Check
$(0 \leq \underline{\mathrm{u}}x \bullet \mathrm{Cr}\, xS \leq \underline{\mathrm{u}}x \bullet \sum y \in B\, \mathrm{Cr}\, x\underline{\mathrm{v}}y)$
and apply 1.78 and 1.114.

Lemma
1.117 $(A \cap B = 0 \wedge s = \sum x \in A \cup B\underline{\mathrm{u}}x \in \mathrm{spl} \to s = \sum x \in A\underline{\mathrm{u}}x + \sum x \in B\underline{\mathrm{u}}x)$
Proof:
Let
$(R = \mathsf{E}\, y, t(y = 0 \wedge t \in A \vee y = 1 \wedge t \in B))$
and note that
$(\mathrm{vs}\, R0 = A \wedge \mathrm{vs}\, R1 = B \wedge$
$\bigwedge y \in 2 \bigwedge n \in 2(y \neq n \to \mathrm{vs}\, Ry \cap \mathrm{vs}\, Rn = 0) \wedge$
$\bigvee y \in 2\, \mathrm{vs}\, Ry = A \cup B)$
WIth the help of 1.115, 1.101, and 1.102 we conclude
$(\mathrm{spl} \ni s$
$= \sum x \in A \cup B\underline{\mathrm{u}}x$
$= \sum y \in 2 \sum x \in \mathrm{vs}\, Ry\underline{\mathrm{u}}x$
$= \sum y \in \mathrm{sng}\, 0 \sum x \in \mathrm{vs}\, Ry\underline{\mathrm{u}}x + \sum y \in \mathrm{sng}\, 1 \sum x \in \mathrm{vs}\, Ry\underline{\mathrm{u}}x)$
$= \sum x \in A\underline{\mathrm{u}}x + \sum x \in B\underline{\mathrm{u}}x)$

Theorem
1.118 $(A \cap B = 0 \to \sum x \in A \cup B\underline{\mathrm{u}}x = \sum x \in A\underline{\mathrm{u}}x + \sum x \in B\underline{\mathrm{u}}x)$
Proof:
Use 1.117 and 1.113.

It is now easy to verify

1.119 Theorems

.0 $(\sum x \in A\underline{\mathrm{u}}x + \sum x \in B\underline{\mathrm{u}}x = \sum x \in A \cup B\underline{\mathrm{u}}x + \sum x \in A \cap B\underline{\mathrm{u}}x)$

.1 $(\bigwedge x \in A \cup B(\underline{\mathrm{u}}x \geq 0) \rightarrow \sum x \in A \cup B \leq \sum x \in A\underline{\mathrm{u}}x + \sum x \in B\underline{\mathrm{u}}x)$

Summation by Finite Partition

Theorem

1.120 $(B \in \mathrm{fnt} \wedge S = \bigvee y \in B\underline{\mathrm{v}}y \wedge \bigwedge y \in B \bigwedge n \in B(y \neq n \rightarrow \underline{\mathrm{v}}y \cap \underline{\mathrm{v}}n = 0)$
$\rightarrow \sum x \in S\underline{\mathrm{u}}x = \sum y \in B \sum x \in \underline{\mathrm{v}}y\underline{\mathrm{u}}x)$

Positive Summation by Partition

Theorem

1.121 $(S = \bigvee y \in B\underline{\mathrm{v}}y \wedge \bigwedge y \in B \bigwedge n \in B(y \neq n \rightarrow \underline{\mathrm{v}}y \cap \underline{\mathrm{v}}n = 0) \wedge \bigwedge x \in S \,\mathrm{rilp}\, \underline{\mathrm{u}}x$
$\rightarrow \sum x \in S\underline{\mathrm{u}}x = \sum y \in B \sum x \in \underline{\mathrm{v}}y\underline{\mathrm{u}}x)$

Theorem

1.122 $(\mathrm{pwr}' A = \sum t \,\mathrm{Cr}\, tA)$

Lemma

1.123 $(\mathrm{function\,is}\, \xi \wedge a = \sum t \in \mathrm{dmn}\, \xi\underline{\mathrm{u}}\,.\,\xi t \wedge b = \sum x \in \mathrm{rng}\, \xi(\underline{\mathrm{u}}x \bullet \mathrm{pwr}' \,\mathrm{hs}\, \xi x) \wedge (a \in \mathrm{spl} \vee b \in \mathrm{spl})$
$\rightarrow a = b)$

Proof:

Assume

$\bigwedge t \bigwedge x(\underline{\mathrm{v}}'tx = (.\,\xi t = x \in \mathrm{U} \wedge 1))$

and note that

$(\underline{\mathrm{v}}'tx = \mathrm{Cr}\, t \,\mathrm{hs}\, \xi x \wedge \sum t\underline{\mathrm{v}}'tx = \mathrm{pwr}' \,\mathrm{hs}\, \xi x \wedge \sum x\underline{\mathrm{v}}'tx = \mathrm{Cr}\, t \,\mathrm{dmn}\, \xi \wedge$
$\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'tx = \underline{\mathrm{u}}\,.\,\xi t \bullet \underline{\mathrm{v}}'tx)$

and recall 1.109 and 1.91.1.

On the one hand

$$\begin{aligned}(\mathrm{spl} \ni a \rightarrow a &= \textstyle\sum t \in \mathrm{dmn}\, \xi\underline{\mathrm{u}}\,.\,\xi t \\ &= \textstyle\sum t(\mathrm{Cr}\, t \,\mathrm{dmn}\, \xi \bullet \underline{\mathrm{u}}\,.\,\xi t) \\ &= \textstyle\sum t(\underline{\mathrm{u}}\,.\,\xi t \bullet \sum x\underline{\mathrm{v}}'tx) \\ &= \textstyle\sum x \sum t(\underline{\mathrm{u}}\,.\,\xi t \bullet \underline{\mathrm{v}}'tx) \\ &= \textstyle\sum x \sum t(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'tx) \\ &= \textstyle\sum x(\underline{\mathrm{u}}x \bullet \mathrm{pwr}' \,\mathrm{hs}\, \xi x) \\ &= \textstyle\sum x \in \mathrm{rng}\, \xi(\underline{\mathrm{u}}x \bullet \mathrm{pwr}' \,\mathrm{hs}\, \xi x) \\ &= b)\end{aligned}$$

On the other hand

$$\begin{aligned}(\mathrm{spl} \ni b \rightarrow b &= \textstyle\sum x \in \mathrm{rng}\, \xi(\underline{\mathrm{u}}x \bullet \mathrm{pwr}' \,\mathrm{hs}\, \xi x) \\ &= \textstyle\sum x(\underline{\mathrm{u}}x \bullet \sum t\underline{\mathrm{v}}'tx) \\ &= \textstyle\sum t \sum x(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'tx) \\ &= \textstyle\sum t \sum x(\underline{\mathrm{u}}\,.\,\xi t \bullet \underline{\mathrm{v}}'tx) \\ &= \textstyle\sum t(\underline{\mathrm{u}}\,.\,\xi t \bullet \sum x\underline{\mathrm{v}}'tx) \\ &= \textstyle\sum t \in \mathrm{dmn}\, \xi\underline{\mathrm{u}}\,.\,\xi t)\end{aligned}$$

From 1.123 we infer

Summation by Substitutiion

Theorem

1.124 $(\text{function is}\,\xi \rightarrow \sum t \in \text{dmn}\,\xi\underline{\text{u}}\,.\,\xi t = \sum x \in \text{rng}\,\xi(\underline{\text{u}}x \bullet \text{pwr}'\,\text{hs}\,\xi x))$

Easy now is

Summation by Transplantation

Theorem

1.125 $(\text{univalent is}\,\xi \rightarrow \sum x \in \text{dmn}\,\xi\underline{\text{u}}\,.\,\xi x = \sum x \in \text{rng}\,\xi\underline{\text{u}}x)$

1.126 Definitions

.0 $(\curlywedge x, y \in A\underline{\text{u}}'xy \equiv \mathsf{E}\, z, w \bigvee x \bigvee y(z = x, y \in A \wedge w = \underline{\text{u}}'xy))$

.1 $(\curlywedge x, y\underline{\text{u}}'xy \equiv \curlywedge x, y \in \text{sqr}\,\text{U}\underline{\text{u}}'xy)$

Theorem

1.127 $(g = \curlywedge x, y \in A\underline{\text{u}}'xy \rightarrow \text{dmn}\,g = \sum x, y \in A(\underline{\text{u}}'xy \in \text{U}) \wedge \bigwedge x \bigwedge y(x, y \in \text{dmn}\,g \rightarrow .\, g(x, y) = \underline{\text{u}}'xy))$

1.128 Definitions

.0 $(\sum x, y \in A\underline{\text{u}}'xy \equiv \sum z \in A\,.\curlywedge x, y \in A\underline{\text{u}}'xyz)$

.1 $(\sum x, y\underline{\text{u}}'xy \equiv \sum x, y \in \text{sqr}\,\text{U}\underline{\text{u}}'xy)$

1.129 Theorems

.0 $(\bigwedge x \bigwedge y(x, y \in A \rightarrow \underline{\text{u}}'xy = \underline{\text{v}}'xy) \rightarrow \sum x, y \in A\underline{\text{u}}'xy = \sum x, y \in A\underline{\text{v}}'xy)$

.1 $(\sim \text{relation is}\,A \rightarrow \sum x, y \in A\underline{\text{u}}'xy = \text{U})$

.2 $(\text{relation is}\,A \wedge \bigwedge x \bigwedge y(\underline{\text{u}}'xy \geq 0) \rightarrow \sum x, y \in A\underline{\text{u}}'xy \geq 0)$

.3 $(\sum x, y \in A\underline{\text{u}}'xy \in \text{kf} \leftrightarrow \sum x, y \in A|\underline{\text{u}}'xy| < \infty)$

.4 $(R = \text{rct}\,AB \wedge s = \sum x, y \in R\underline{\text{u}}'xy \in \text{spl} \rightarrow \sum x \in A \sum y \in B\underline{\text{u}}'xy = s = \sum y \in B \sum x \in A\underline{\text{u}}'xy)$

Proof:

Let

$(g = \curlywedge x, y\underline{\text{u}}'xy)$

and check that

$(\text{spl} \ni s$

$= \sum x, y \in R\underline{\text{u}}'xy$

$= \sum z \in R\,.\,gz$

$= \sum z \in \bigvee xinA \bigvee y \in B\,\text{sng}(x, y)\,.\,gz$

$= \sum x \in A \sum z \in \bigvee y \in B\,\text{sng}(x, y)\,.\,gz$

$= \sum x \in A \sum y \in B \sum z \in \text{sng}(x, y)\,.\,gz$

$= \sum x \in A \sum y \in B\,.\,g(x, y)$

$= \sum x \in A \sum y \in B\underline{\text{u}}'xy)$.

Similarly

$(s = \sum y \in B \sum x \in A\underline{\text{u}}'xy)$.

From .2, .3, and .4 we infer

Summation by Commutation

Theorem

1.130 $(\sum x \in A \sum y \in B|\underline{\text{u}}'xy| < \infty \rightarrow \sum x \in A \sum y \in B\underline{\text{u}}'xy = \sum y \in B \sum x \in A\underline{\text{u}}'xy \in \text{kf})$

With the help of 1.95.19 we now easily check

Dominated Summation by Distribution

Theorem

1.131 $(\sum x \in A(|\underline{\mathrm{u}}x| \bullet \sum y \in B|\underline{\mathrm{v}}'xy|) < \infty \wedge s = \sum x \in A(\underline{\mathrm{u}}x \bullet \sum y \in B\underline{\mathrm{v}}'xy)$
$\quad \rightarrow \sum x \in A \sum y \in B(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy) = s = \sum y \in B \sum x \in B(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy) \in \mathrm{kf})$

The next theorem, which we shall not use, casts light on 1.109.

Theorem

1.132 $(\bigwedge x \bigwedge y(\underline{\mathrm{v}}'xy \geq 0) \rightarrow \sum x(\underline{\mathrm{u}}x \bullet \sum y\underline{\mathrm{v}}'xy) = \sum x, y(\underline{\mathrm{u}}x \bullet \underline{\mathrm{v}}'xy))$

Hint. Use 1.108 and 1.107.

Perhaps of interest are 1.133 and 1.134 below

Theorem

1.133 $(\mathrm{lm}\, \alpha\, \mathrm{sumrun} \sum x \in \alpha\underline{\mathrm{u}}x = a \in \mathrm{kf} \rightarrow a = \sum x\underline{\mathrm{u}}x)$

1.134 $(\sum x \in A\underline{\mathrm{u}}x = \sum x \in A\, \mathrm{prt}'\, \underline{\mathrm{u}}x + \sum x \in A(\mathrm{i} \cdot \mathrm{prt}''\, \underline{\mathrm{u}}x))$

Definition

1.135 $(\mathrm{nt}\, ab \equiv \mathsf{E}\, t(a \leq t \leq t \vee b \leq t \leq a))$

Definition

1.136 $(\not\sum x \in A\underline{\mathrm{u}}x \equiv \mathrm{lin}\, n \sum x \in A \cap \mathrm{nt} - nn\underline{\mathrm{u}}x)$

Note that $(\not\sum j \in \omega' = 0)$ because of the symmetry employed in 1.136. Such symmetry is convenient for some purposes.

1.137 Theorems

.0 $(\not\sum j \in \omega\underline{\mathrm{u}}j = \mathrm{lin}\, n \sum j \in n\underline{\mathrm{u}}j)$
.1 $(\sum n \in \omega\underline{\mathrm{u}}n \in \mathrm{spl} \rightarrow \not\sum n \in \omega\underline{\mathrm{u}}n = \sum n \in \omega\underline{\mathrm{u}}n)$
.2 $(\not\sum n \in \omega|\underline{\mathrm{u}}n| = \sum n \in \omega|\underline{\mathrm{u}}n|)$
.3 $(\not\sum n \in \omega < \infty \not\sum n \in \omega\underline{\mathrm{u}}n = \sum n \in \omega\underline{\mathrm{u}}n)$

1.138 Exercises

.0 $(0 \leq \epsilon \wedge n \in \omega \rightarrow {}^{\centerdot}(1 + \epsilon)n \geq 1 + n \bullet \epsilon)$
.1 $(1 < y \rightarrow \mathrm{lin}\, n^{\centerdot}yn = \infty)$
.2 $(|x| < 1 \rightarrow 0 = \mathrm{lin}\, n^{\centerdot}|x|n = \mathrm{lin}\, n^{\centerdot}xn)$
.3 $(|x| < \infty \wedge n \in \omega \rightarrow (1 - x) \cdot \sum j \in n^{\centerdot}xj = 1 - {}^{\centerdot}xn)$
.4 $(|x| < 1 \rightarrow \sum j \in \omega^{\centerdot}xj = 1/(1 - x))$

1.139 Exercise

Let us agree that

$((\xi \,\mathrm{is\,diadic\,for}\, x) \leftrightarrow \bigvee n \in \omega(\xi \in \mathrm{On}\, n \,\mathrm{To}\, 2 \wedge$
$0 \le x — \sum j \in n(\centerdot \xi j \cdot {}^{\centerdot}2 - j) < {}^{\centerdot}2 - (n+1)))$.

Let us also agree that

$(\mathrm{diad}\, x \equiv \mathsf{E}\, \xi(\xi \,\mathrm{is\,diadic\,for}\, x))$

and that

$(\mathrm{Dc}\, x \equiv \nabla\, \mathrm{diad}\, x)$.

Show that

$(x \in \mathrm{nt}\, 01 \rightarrow \mathrm{nest\,is\,diad}\, x \wedge \mathrm{Dc}\, x \in \mathrm{On}\, \omega \,\mathrm{To}\, 2 \wedge x = \sum j \in \omega(\centerdot \mathrm{Dc}\, xj \cdot {}^{\centerdot}2 - j))$

7. Integration

According to Riemann an integral is a limit of a sum. Using sums very similar to those considered by Riemann we seek and find a definition of integral which even in its special manifestations in 7.24.0 is flexible enough to embody the Riemann-Stieltjes integral on the one hand and the Lebesgue integral on the other. Our general attack is susceptible of a variety of specializations and easy generalizations.

General Methods

Riemann sums suggest

7.0 Definitions

.0 $(\mathrm{rsum}\, f\xi\varphi \equiv \sum \beta \in \mathrm{dmn}\, \xi(\boldsymbol{.}\, f\, \boldsymbol{.}\, \xi\beta \bullet \boldsymbol{.}\, \varphi\beta))$

.1 $(\int\# \, Mf\varphi\} \equiv \mathrm{lm}\, \xi M\, \mathrm{rsum}\, f\xi\varphi)$

.2 $(\overline{\int}\# \, Mf\varphi\} \equiv \overline{\mathrm{lm}}\, \xi M\, \mathrm{rsum}\, f\xi\varphi)$

.3 $(\underline{\int}\# \, Mf\varphi\} \equiv \underline{\mathrm{lm}}\, \xi M\, \mathrm{rsum}\, f\xi\varphi)$

In the above we regard f as a numerical valued function to be integrated, ξ as a generalized Riemann interpolant, φ as a numerical valued set function, possibly a measure, with respect to which the function f is to be integrated, and the run M as a method of integration.

The set sweep M, over which we integrate, is described in 7.1.0. The unimportant subsets of sweep M are described in 7.1.1.

7.1 Definitions

.0 $(\mathrm{sweep}\, M \equiv \bigvee \xi \in \mathrm{rng}\, M \bigvee \beta \in \mathrm{dmn}\, \xi\, \mathrm{sng}\, \boldsymbol{.}\, \xi\beta)$

.1 $(\mathrm{ignore\#}\, M\varphi A \equiv (A \subset \mathrm{sweep}\, M \wedge \mathrm{far}\, M\xi \bigwedge \beta \in \mathrm{dmn}\, \xi(\boldsymbol{.}\, \xi\beta \in A \rightarrow \boldsymbol{.}\, \varphi\beta = 0)))$

Definition

7.2 $(\mathrm{gauge} \equiv \mathrm{To} \sqsubseteq x(0 \le x \le \infty))$

7.3 Definitions

.0 $(\mathrm{alm\#}\, M\varphi x \underline{\mathrm{u}}x \equiv \mathrm{ignore\#}\, M\varphi(\mathrm{sweep}\, M \sim \sqsubseteq x \underline{\mathrm{u}}x))$

.1 $(\mathrm{almostall\, by}\, M\varphi x \underline{\mathrm{u}}x \equiv \mathrm{alm\#}\, M\varphi x \underline{\mathrm{u}}x)$

7.4 Theorems

.0 $(A \subset B \wedge \mathrm{ignore\#}\, M\varphi B \rightarrow \mathrm{ignore\#}\, M\varphi A)$

.1 $(\mathrm{ignore\#}\, M\varphi A \wedge \mathrm{ignore\#}\, M\varphi B \leftrightarrow \mathrm{ignore\#}\, M\varphi(A \cup B))$

.2 $(\mathrm{run\, is}\, M \rightarrow \mathrm{ignore\#}\, M\varphi 0)$

.3 $(\mathrm{ignore\#}\, M\varphi(\mathrm{sweep}\, M \sim A) \leftrightarrow \mathrm{far}\, M\xi \bigwedge \beta \in \mathrm{dmn}\, \xi(\boldsymbol{.}\, \xi\beta \in A \vee \boldsymbol{.}\, \varphi\beta = 0))$

.4 $(\mathrm{run\, is}\, M \wedge \bigwedge x \in \mathrm{sweep}\, M \underline{\mathrm{u}}x \rightarrow \mathrm{alm\#}\, M\varphi x \underline{\mathrm{u}}x)$

.5 $(\mathrm{run\, is}\, M \wedge y \rightarrow \mathrm{alm\#}\, M\varphi xy)$

.6 $(\mathrm{alm\#}\, M\varphi x(\underline{\mathrm{u}}x \rightarrow \underline{\mathrm{v}}x) \rightarrow (\mathrm{alm\#}\, M\varphi x \underline{\mathrm{u}}x \rightarrow \mathrm{alm\#}\, M\varphi x \underline{\mathrm{v}}x))$

.7 $(\mathrm{alm\#}\, M\varphi x(\underline{\mathrm{u}}x \wedge \underline{\mathrm{v}}x) \leftrightarrow \mathrm{alm\#}\, M\varphi x \underline{\mathrm{u}}x \wedge \mathrm{alm\#}\, M\varphi x \underline{\mathrm{v}}x)$

7.5 Lemmas

.0 $(\beta \in \mathrm{dmn}\,\xi \leftrightarrow \mathbf{.}\,\xi\beta \in \mathrm{U} \leftrightarrow \mathbf{.}\,\xi\beta \neq \mathrm{U})$

.1 $(g = \curlywedge x\,\mathbf{.}\,fx \rightarrow \mathrm{dmn}\,g = \mathrm{dmn}\,f \wedge \mathbf{.}\,gx = \mathbf{.}\,fx)$

.2 $(f = \curlywedge x\underline{\mathrm{u}}x \wedge g = \curlywedge x\underline{\mathrm{v}}x \wedge h = \curlywedge x(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x) \rightarrow \mathbf{.}\,hx = \mathbf{.}\,fx + \mathbf{.}\,gx \wedge h = \curlywedge x(\mathbf{.}\,fx + \mathbf{.}\,gx))$

.3 $(h = \curlywedge x(\mathbf{.}\,fx + \mathbf{.}\,gx) \rightarrow \mathbf{.}\,hx = \mathbf{.}\,fx + \mathbf{.}\,gx)$

.4 $(f = \curlywedge x\underline{\mathrm{u}}x \wedge g = \curlywedge x(c \cdot \underline{\mathrm{u}}x) \rightarrow \mathbf{.}\,gx = c \cdot \mathbf{.}\,fx \wedge g = \curlywedge x(c \cdot \mathbf{.}\,fx))$

.5 $(g = \curlywedge x(c \cdot \mathbf{.}\,fx) \rightarrow \mathbf{.}\,gx = c \cdot \mathbf{.}\,fx)$

.6 $(f = \curlywedge x\underline{\mathrm{u}}x \wedge g = \curlywedge x\,\mathrm{prt}'\,\underline{\mathrm{u}}x \wedge h = \curlywedge x\,\mathrm{prt}''\,\underline{\mathrm{u}}x \rightarrow \mathrm{prt}'\,\mathbf{.}\,fx = \mathbf{.}\,gx \wedge \mathrm{prt}''\,\mathbf{.}\,fx = \mathbf{.}\,hx)$

.7 $(f = \curlywedge x\underline{\mathrm{u}}x \wedge g = \curlywedge x|\underline{\mathrm{u}}x| \rightarrow \bigwedge x \in \mathrm{U}(\mathbf{.}\,gx = |\mathbf{.}\,fx|))$

.8 $(f = \curlywedge x\underline{\mathrm{u}}x \wedge g = \curlywedge x\underline{\mathrm{v}}x \wedge \underline{\mathrm{u}}x = \underline{\mathrm{v}}x \rightarrow \mathbf{.}\,fx = \mathbf{.}\,gx)$

.9 $(f = \curlywedge x\underline{\mathrm{u}}x \rightarrow \bigwedge x \in \mathrm{U}(\mathbf{.}\,fx \in A \leftrightarrow \underline{\mathrm{u}}x \in A))$

7.6 Theorems

.0 $(\int\# Mf\varphi\} \neq \mathrm{U} \rightarrow \mathrm{run\,is}\,M \wedge \int\# Mf\varphi\} \in \mathrm{spl})$

.1 $(g = \curlywedge x\,\mathbf{.}\,fx \rightarrow \int\# Mg\varphi\} = \int\# Mf\varphi\})$

.2 $(\mathrm{alm}\# M\varphi x(\mathbf{.}\,fx = \mathbf{.}\,gx) \rightarrow \int\# Mf\varphi\} = \int\# Mg\varphi\})$

7.7 Theorems

.0 $(J = \int\# Mf\varphi\} + \int\# Mg\varphi\} \in \mathrm{U} \wedge h = \curlywedge x(\mathbf{.}\,fx + \mathbf{.}\,gx) \rightarrow J = \int\# Mh\varphi\})$

.1 $(0 \neq c \in \mathrm{rf} \wedge g = \curlywedge x(c \cdot \mathbf{.}\,fx) \rightarrow \int\# Mg\varphi\} = c \cdot \int\# Mf\varphi\}$

.2 $(g = \curlywedge x(c \cdot \mathbf{.}\,fx) \wedge J = c \cdot \int\# Mf\varphi\} \in \mathrm{kf} \rightarrow J = \int\# Mg\varphi\})$

.3 $(\mathrm{run\,is}\,M \rightarrow \int\# M \curlywedge x0\varphi\} = 0)$

.4 $(\int\# Mf\varphi\} \in \mathrm{spl} \rightarrow \mathrm{alm}\# M\varphi x(\mathbf{.}\,fx \in \mathrm{spl}))$

.5 $(\int\# Mf\varphi\} \in \mathrm{kf} \rightarrow \mathrm{alm}\# M\varphi x(\mathbf{.}\,fx \in \mathrm{kf}))$

A theory more attractive, in a number of respects, emerges if the integral in 7.0.1 is narrowed somewhat in the infinite cases. This is done in

7.8 Definitions

.0 $(\mathrm{neared}\# M\varphi\,\mathrm{by}\,R\,\mathrm{is}\,f \equiv (\bigwedge n \in \omega(\int\# M\,\mathrm{vs}\,Rn\varphi\} \in \mathrm{kf}) \wedge \mathrm{alm}\# M\varphi x(\mathrm{lin}\,n\,\mathbf{.}\,\mathrm{vs}\,Rnx = \mathbf{.}\,fx)))$

.1 $(\mathrm{neared}\# M\varphi f \equiv \bigvee R\,\mathrm{neared}\# M\varphi\,\mathrm{by}\,R\,\mathrm{is}\,f)$

.2 $(\int\# Mf\varphi] \equiv (\mathrm{neared}\# M\varphi f \rightarrow \int\# Mf\varphi\}))$

.3 $(\int\# M\underline{\mathrm{u}}x\varphi\,\mathrm{d}x \equiv \int\# M \curlywedge x\underline{\mathrm{u}}x\varphi])$

.4 $(\int\# MA;\underline{\mathrm{u}}x\varphi\,\mathrm{d}x \equiv \int\# M(\mathrm{Cr}\,xA \bullet \underline{\mathrm{u}}x)\varphi\,\mathrm{d}x)$

In the above we regard R as a relation whose vertical sections are functions approximating the function f.

7.9 Lemmas

.0 $(\mathrm{alm}\# M\varphi x(\mathbf{.}\,fx = \mathbf{.}\,gx) \rightarrow \mathrm{neared}\# M\varphi\,\mathrm{by}\,R\,\mathrm{is}\,f \leftrightarrow \mathrm{neared}\# M\varphi\,\mathrm{by}\,R\,\mathrm{is}\,g)$

.1 $(h = \curlywedge x(\mathbf{.}\,fx + \mathbf{.}\,gx) \wedge \mathrm{alm}\# M\varphi x(\mathbf{.}\,hx \in \mathrm{U}) \wedge \mathrm{neared}\# M\varphi\,\mathrm{by}\,R\,\mathrm{is}\,f \wedge \mathrm{neared}\# M\varphi\,\mathrm{by}\,S\,\mathrm{is}\,g\,\wedge$

$T = \mathsf{E}\,n,t(t \in \curlywedge x(\mathbf{.}\,\mathrm{vs}\,Rnx + \mathbf{.}\,\mathrm{vs}\,Snx))$

$\rightarrow \mathrm{neared}\# M\varphi\,\mathrm{by}\,T\,\mathrm{is}\,h)$

Proof:

Note first

$$(n \in \omega \rightarrow \mathrm{vs}\,Tn = \curlywedge x(\mathbf{.}\,\mathrm{vs}\,Rnx + \mathbf{.}\,\mathrm{vs}\,Snx))\ .$$

Hence, because of 7.7.0,

$(n \in \omega \rightarrow \mathrm{kf} \ni \int\# M \,\mathrm{vs}\, Rn\varphi\} + \int\# M \,\mathrm{vs}\, Sn\varphi\} = \int\# M \,\mathrm{vs}\, Tn\varphi\})$

and

$\bigwedge n \in \omega(\int\# M \,\mathrm{vs}\, Tn\varphi\} \in \mathrm{kf})$.

Also, because of 7.5.3,

$\mathrm{alm}\# M\varphi x(\mathrm{U} \ni \centerdot hx$

$= \centerdot fx + \centerdot gx$

$= \mathrm{lin}\, n \centerdot \mathrm{vs}\, Rnx + \mathrm{lin}\, n \centerdot \mathrm{vs}\, Snx$

$= \mathrm{lin}\, n(\centerdot \mathrm{vs}\, Rnx + \centerdot \mathrm{vs}\, Snx)$

$= \mathrm{lin}\, n \centerdot \mathrm{vs}\, Tnx)$

and

$\mathrm{alm}\# M\varphi x(\mathrm{lin}\, n \centerdot \mathrm{vs}\, Tnx = \centerdot hx)$.

The desired conclusion is at hand.

.2 $(0 \neq c \in \mathrm{rf} \wedge g = \curlywedge x(c \cdot \centerdot fx) \wedge S = \sqsubseteq n, t(t \in \curlywedge x(c \cdot \centerdot \mathrm{vs}\, Rnx))$
$\rightarrow \mathrm{neared}\# M\varphi \,\mathrm{by}\, R \,\mathrm{is}\, f \leftrightarrow \mathrm{neared}\# M\varphi \,\mathrm{by}\, S \,\mathrm{is}\, g)$

.3 $(\int\# Mf\varphi\} \in \mathrm{kf} \wedge R = \sqsubseteq n, t(t \in f) \rightarrow \mathrm{neared}\# M\varphi \,\mathrm{by}\, R \,\mathrm{is}\, f)$

.4 $(\mathrm{neared}\# M\varphi \,\mathrm{by}\, R \,\mathrm{is}\, f \rightarrow \mathrm{alm}\# M\varphi x(\centerdot fx \in \mathrm{spl}))$

Theorem

7.11 $(\int\# Mf\varphi] \neq \mathrm{U} \rightarrow$

.0 $\mathrm{run\,is}\, M \wedge$

.1 $\mathrm{neared}\# M\varphi f \wedge$

.2 $\int\# Mf\varphi] = \int\# Mf\varphi\} \in \mathrm{spl})$

7.12 Theorems

.0 $(\mathrm{neared}\# M\varphi f \rightarrow \int\# Mf\varphi] = \int\# Mf\varphi\})$

.1 $(\int\# Mf\varphi\} \in \mathrm{kf} \rightarrow \int\# Mf\varphi] = \int\# Mf\varphi\})$

.2 $(\mathrm{alm}\# M\varphi x(\centerdot fx = \centerdot gx) \rightarrow \int\# Mf\varphi] = \int\# Mg\varphi])$

.3 $(g = \curlywedge x \centerdot fx \rightarrow \int\# Mg\varphi] = \int\# Mf\varphi])$

.4 $(J = \int\# Mf\varphi] + \int\# Mg\varphi] \wedge h = \curlywedge x(\centerdot fx + \centerdot gx) \rightarrow J = \int\# Mg\varphi])$

Proof:

According to 7.11.2, 7.7.0, and 7.7.4

$(\int\# Mh\varphi\} = J \in \mathrm{U} \wedge \mathrm{alm}\# M\varphi x(\centerdot hx \in \mathrm{spl}))$.

Because of this, 7.11.1, and 7.10.1

$\mathrm{neared}\# M\varphi h$.

Hence

$(\int\# Mh\varphi] = \int\# Mh\varphi\} = J)$

.5 $(0 \neq c \in \mathrm{rf} \wedge g = \curlywedge x(c \cdot \centerdot fx) \rightarrow \int\# Mg\varphi] = c \cdot \int\# Mf\varphi])$

.6 $(g = \curlywedge x(c \cdot \centerdot fx) \wedge J = c \cdot \int\# Mf\varphi] \in \mathrm{kf} \rightarrow J = \int\# Mg\varphi])$

.7 $(\mathrm{run\,is}\, M \rightarrow \int\# M \curlywedge x0\varphi] = 0)$

.8 $(\int\# Mf\varphi] \in \mathrm{spl} \rightarrow \mathrm{alm}\# M\varphi x(\centerdot fx \in \mathrm{spl}))$

.9 $(\int\# Mf\varphi] \in \mathrm{kf} \rightarrow \mathrm{alm}\# M\varphi x(\centerdot fx \in \mathrm{kf}))$

Theorem
7.13 $(f = ⋌ x\underline{\mathrm{u}}x \wedge \int\# M\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x \neq \mathrm{U} \rightarrow$
.0 $\mathrm{run\,is}\, M \wedge$
.1 $\mathrm{neared}\# M\varphi f \wedge$
.2 $\int\# M\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x = \int\# Mf\varphi\} \in \mathrm{spl} \wedge$
.3 $\mathrm{alm}\# M\varphi x(\boldsymbol{.}\, fx = \underline{\mathrm{u}}x \in \mathrm{spl}))$

7.14 Theorems
.0 $(f = ⋌ x\underline{\mathrm{u}}x \wedge \mathrm{neared}\# M\varphi f \rightarrow \int\# M\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x = \int\# Mf\varphi\})$
.1 $(f = ⋌ x\underline{\mathrm{u}}x \wedge \int\# Mf\varphi\} \in \mathrm{kf} \rightarrow \int\# M\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x = \int\# Mf\varphi\})$

Taking advantage of 7.5 and 7.12 we rather easily check
7.15 Theorems
.0 $(\int\# M\,\boldsymbol{.}\, fx\varphi \,\mathrm{d}\,x = \int\# Mf\varphi])$
.1 $(\mathrm{alm}\# M\varphi x(\underline{\mathrm{u}}x = \underline{\mathrm{v}}x) \rightarrow \int\# M\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x = \int\# M\underline{\mathrm{v}}x\varphi \,\mathrm{d}\,x)$
.2 $(J = \int\# M\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x + \int\# M\underline{\mathrm{v}}x\varphi \,\mathrm{d}\,x \in \mathrm{U} \rightarrow J = \int\# M(\underline{\mathrm{u}}x + \underline{\mathrm{v}}x)\varphi \,\mathrm{d}\,x)$
.3 $(0 \neq c \in \mathrm{rf} \rightarrow \int\# M(c \cdot \underline{\mathrm{u}}x)\varphi \,\mathrm{d}\,x = c \cdot \int\# M\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x)$
.4 $(J = c \cdot \int\# M\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x \in \mathrm{kf} \rightarrow J = \int\# M(c \cdot \underline{\mathrm{u}}x)\varphi \,\mathrm{d}\,x)$
.5 $(\mathrm{run\,is}\, M \rightarrow \int\# M0\varphi \,\mathrm{d}\,x = 0)$
.6 $(\int\# M\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x \in \mathrm{spl} \rightarrow \mathrm{alm}\# M\varphi x(\underline{\mathrm{u}}x \in \mathrm{spl}))$
.7 $(\int\# M\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x \in \mathrm{kf} \rightarrow \mathrm{alm}\# M\varphi x(\underline{\mathrm{u}}x \in \mathrm{kf}))$

7.16 Theorems
.0 $(\varphi \in \mathrm{To\,rl} \wedge J = \int\# M\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x \in \mathrm{kf} \rightarrow J = \int\# M\,\mathrm{prt}'\,\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x + \mathrm{i} \cdot \int\# M\,\mathrm{prt}''\,\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x)$
.1 $(\varphi \in \mathrm{To\,rl} \wedge J = \int\# M\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x \in \mathrm{spl} \rightarrow \mathrm{prt}'\,J = \int\# M\,\mathrm{prt}'\,\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x)$
.2 $(\varphi \in \mathrm{To\,rl} \wedge \mathrm{alm}\# M\varphi x(\underline{\mathrm{u}}x \in \mathrm{rl}) \rightarrow \int\# M\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x \backsim\in \sim\mathrm{rl})$

7.17 Theorems
.0 $(\varphi \in \mathrm{gauge} \wedge \mathrm{alm}\# M\varphi x(\underline{\mathrm{u}}x \geq 0) \rightarrow \int\# M\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x \backsim< 0)$
.1 $(\varphi \in \mathrm{gauge} \wedge \mathrm{alm}\# M\varphi x(\underline{\mathrm{u}}x \leq \underline{\mathrm{v}}x) \rightarrow \int\# M\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x \backsim> \int\# M\underline{\mathrm{v}}x\varphi \,\mathrm{d}\,x)$
.2 $(\varphi \in \mathrm{gauge} \wedge \mathrm{alm}\# M\varphi x(0 \leq \underline{\mathrm{u}}x \leq \underline{\mathrm{v}}x) \wedge \int\# M\underline{\mathrm{v}}x\varphi \,\mathrm{d}\,x = 0 \rightarrow \int\# M\underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x = 0)$

Integration by Refinement

7.18 Definitions
- .0 (selector $\equiv \mathsf{E}\, \xi \in$ To U $\bigwedge \beta \in$ dmn $\xi(\centerdot\, \xi\beta \in \beta))$
- .1 (partion $A \equiv \mathsf{E}\, D \in$ cbl $\cap$ dsjn $\cap$ sb $\sim 1(\nabla D = A))$
- .2 (grator $\equiv \mathsf{E}\, \varphi \in$ To spl$(\centerdot\, \varphi 0 = 0))$
- .3 $(\mathrm{sng}'\, x \equiv (\sim 1 \cap \mathrm{sng}\, x))$
- .4 $((D \sqcap S) \equiv \bigvee x \in D \bigvee y \in S\, \mathrm{sng}'(x \cap y))$
- .5 $((S \subset\subset D) \equiv \bigwedge x \in S \bigvee y \in D(x \subset y))$

7.19 Definitions
- .0 (scheme $\varphi \equiv \mathsf{E}\, D \in$ partion rlm $\varphi \wedge \varphi \in$ grator $\bigwedge T \in$ dmn $\varphi(\centerdot\, \varphi T = \sum \beta \in D \centerdot \varphi(T\beta)))$
- .1 (grid $\varphi \equiv \mathsf{E}\, G \in$ scheme $\varphi \bigwedge D \in$ scheme $\varphi(G \sqcap D \in$ scheme $\varphi))$

Definition
7.20 (mode $\varphi \equiv \mathsf{E}\, D, \xi(D \in$ grid $\varphi \wedge \xi \in$ selector $\wedge$ dmn $\xi \in$ grid $\varphi \wedge$ dmn $\xi \subset\subset D))$

Definition
7.21 (schememode $\varphi \equiv \mathsf{E}\, D, \xi(D \in$ grid $\varphi \wedge \xi \in$ selector $\wedge$ dmn $\xi \in$ scheme $\varphi \wedge$ dmn $\xi \subset\subset D))$

We formulate 7.21 merely to indicate one of the alternative approaches. We prefer 7.20 to 7.21 tho much can be done with the latter.

7.22 Definitions
- .0 $(\int f\varphi\} \equiv \int\#$ mode $\varphi f \varphi\})$
- .1 $(\overline{\int} f\varphi\} \equiv \overline{\int}\#$ mode $\varphi f \varphi\})$
- .2 $(\underline{\int} f\varphi\} \equiv \underline{\int}\#$ mode $\varphi f \varphi\})$

Definition
7.23 $(\int f\varphi] \equiv \int\#$ mode $\varphi f \varphi])$

7.24 Definitions
- .0 $(\int \underline{\mathrm{u}} x \varphi\, \mathrm{d}\, x \equiv \int\#$ mode $\varphi \underline{\mathrm{u}} x \varphi\, \mathrm{d}\, x)$
- .1 $(\int A; \underline{\mathrm{u}} x \varphi\, \mathrm{d}\, x \equiv \int\#$ mode $\varphi A; \underline{\mathrm{u}} x \varphi\, \mathrm{d}\, x)$

7.25 Definitions
- .0 (ignore $\varphi A \equiv$ ignore$\#$ mode $\varphi\varphi A)$
- .1 (alm $\varphi x \underline{\mathrm{u}} x \equiv$ alm$\#$ mode $\varphi\varphi x \underline{\mathrm{u}} x)$

7.26 Definitions
- .0 (neared φ by R is $f \equiv$ neared$\#$ mode $\varphi\varphi$ by R is $f)$
- .1 (neared $\varphi f \equiv$ neared$\#$ mode $\varphi\varphi f)$

7.27 Definitions
- .0 (swath $D \equiv$ (selector $\cap$ On $D))$
- .1 (swing $fD\varphi \equiv$ sup $\xi, \eta, \in$ swath $D|$ rsum $f\xi\varphi -$ rsum $f\eta\varphi|)$

7.28 Definitions

.0 $(\mathrm{oscl}\, f\beta \equiv (\beta \neq 0 \wedge \sup x, y, \in \beta | \,.\, fx - \,.\, fy|))$

.1 $(\mathrm{osum}\, fD\varphi \equiv (\inf \xi \in \mathrm{swath}\, D|0 \cdot \mathrm{rsum}\, f\xi\varphi| + \sum \beta \in D| \,\mathrm{oscl}\, f\beta \bullet .\, \varphi\beta|))$

In connection with .0

$(\beta = 0 \rightarrow \mathrm{oscl}\, f\beta = 0)$

and

$(\beta \neq 0 \rightarrow \mathrm{oscl}\, f\beta = \sup x, y, \in \beta | \,.\, fx - \,.\, fy|)$.

In connection with .1

$(\bigvee \xi \in \mathrm{swath}\, D(\mathrm{rsum}\, f\xi\varphi \in \mathrm{kf}) \rightarrow \mathrm{osum}\, fD\varphi = \sum \beta \in D| \,\mathrm{oscl}\, f\beta \bullet .\, \varphi\beta|)$

and

$(\mathrm{osum}\, fD\varphi < \infty \rightarrow \bigvee \xi \in \mathrm{swath}\, D(\mathrm{rsum}\, f\xi\varphi \in \mathrm{kf}))$.

The oscillation integral is described in

Definition

7.29 $(\overline{\underline{\int}} \underline{\mathrm{u}} x\varphi \,\mathrm{d}\, x \equiv \inf D \in \mathrm{grid}\, \varphi\, \mathrm{osum} \curlywedge x \underline{\mathrm{u}} x D\varphi)$

Definition

7.30 $(\mathrm{vrn}\, \varphi \equiv \curlywedge T \in \mathrm{dmn}\, \varphi \sup D \in \mathrm{grid}\, \varphi \sum \beta \in D| \,.\, \varphi(T\beta)|)$

Definition

7.31 $(\mathrm{gridor} \equiv \mathsf{E}\, \varphi \in \mathrm{grator}(\mathrm{grid}\, \varphi = \mathsf{E}\, D \in \mathrm{partion}\, \mathrm{rlm}\, \varphi \bigwedge T \in \mathrm{dmn}\, \varphi \bigwedge \beta \in D(T\beta \in \mathrm{dmn}\, \varphi)))$

7.32 Theorems

.0 $(D \Cap D' = D' \Cap D)$

.1 $(D \Cap (D' \Cap D'') = (D \Cap D') \Cap D'')$

.2 $(D \cup D' \Cap D'' = D \Cap D' \cup D \Cap D'')$

.3 $(\nabla(D \Cap D') = \nabla D \cap \nabla D')$

.4 $(D \Cap D' \subset\subset D \wedge D \Cap D' \subset\subset D'')$

.5 $(D'' \subset\subset D' \subset\subset D \rightarrow D'' \subset\subset D)$

.6 $(D' \subset\subset D \rightarrow D' = \bigvee \alpha \in D(D' \,\mathrm{sb}\, \alpha))$

.7 $(D' \subset\subset D \in \mathrm{dsjn} \wedge \alpha \in D \wedge \alpha' \in D' \wedge \alpha\alpha' \neq 0 \rightarrow \alpha' \subset \alpha)$

.8 $(\alpha\alpha' = 0 \rightarrow \mathrm{sb}\, \alpha\, \mathrm{sb}\, \alpha' = 1)$

.9 $(\alpha\alpha' = 0 \rightarrow \sim\!1\, \mathrm{sb}\, \alpha\, \mathrm{sb}\, \alpha' = 0)$

7.33 Theorems

.0 $(D \in \mathrm{grid}\, \varphi \wedge D' \in \mathrm{grid}\, \varphi \rightarrow D \Cap D' \in \mathrm{grid}\, \varphi)$

.1 $(\varphi \in \mathrm{grator} \rightarrow \mathrm{sng}'\, \mathrm{rlm}\, \varphi \in \mathrm{scheme}\, \varphi)$

.2 $(\varphi \in \mathrm{grator} \rightarrow \mathrm{sng}'\, \mathrm{rlm}\, \varphi \in \mathrm{grid}\, \varphi)$

Although Theorem 7.34 below involves the principle of choice, some interesting instances of do not.

Theorem

7.34 $(\varphi \in \mathrm{grator} \wedge M = \mathrm{mode}\, \varphi \rightarrow \mathrm{dmn}\, M = \mathrm{grid}\, \varphi \wedge \mathrm{sweep}\, M = \mathrm{rlm}\, \varphi \wedge \mathrm{run}\, \mathrm{is}\, M)$

Lemma

7.35 $(.\, \psi 0 = 0 \wedge \alpha \in D \in \mathrm{dsjn} \wedge D' \in \mathrm{dsjn} \wedge D'' = D \Cap D'$

$\rightarrow \sum \alpha'' \in D \cap \mathrm{sb}\, \alpha \,.\, \psi\alpha'' = \sum \alpha' \in D' \,.\, \psi(\alpha\alpha'))$

Proof:

Suppose

$(F' = D' \sqsubseteq \alpha'(\alpha\alpha' \neq 0) \wedge F'' = D'' \,\mathrm{sb}\, \alpha \wedge N = \curlywedge \alpha' \in F'(\alpha\alpha'))$.

Note carefully that

(univalent is $N \wedge \mathrm{dmn}\, N = F' \wedge \mathrm{rng}\, N = F''$)

and use summation by transplantation in checking

$$
\begin{aligned}
&(\textstyle\sum \alpha' \in D' \,.\, \psi(\alpha\alpha') \\
&\quad = \textstyle\sum a \in F' \,.\, \psi(\alpha\alpha') + \sum \alpha' \in D \backsim F' \,.\, \psi(\alpha\alpha') \\
&\quad = \textstyle\sum \alpha' \in F' \,.\, \psi(\alpha\alpha') \\
&\quad = \textstyle\sum \alpha' \in F' \,.\, \psi \,.\, N\alpha' \\
&\quad = \textstyle\sum \alpha'' \in F'' \,.\, \psi\alpha'' \\
&\quad = \textstyle\sum \alpha'' \in D'' \cap \mathrm{sb}\, \alpha \,.\, \psi\alpha'')\ .
\end{aligned}
$$

Theorem

7.36 ($\mathrm{fnt\,scheme}\, \varphi \subset \mathrm{grid}\, \varphi$)

Proof:

With the help of 7.35, 7.32, and summation by finite partition we infer

$$
\begin{aligned}
&(D \in \mathrm{fnt\,scheme}\, \varphi \wedge D' \in \mathrm{scheme}\, \varphi \wedge D'' = D \sqcap D'' \wedge T \in \mathrm{dmn}\, \varphi \\
&\quad \to \mathrm{spl} \ni .\, \varphi T \\
&\qquad = \textstyle\sum \alpha \in D \,.\, \varphi(T\alpha) \\
&\qquad = \textstyle\sum \alpha \in D \sum \alpha' \in D' \,.\, \varphi(T\alpha\alpha') \\
&\qquad = \textstyle\sum \alpha \in D \sum \alpha'' \in D'' \cap \mathrm{sb}\, \alpha \,.\, \varphi(T\alpha'') \\
&\qquad = \textstyle\sum \alpha'' \in \bigvee \alpha \in D(D'' \cap \mathrm{sb}\, \alpha) \,.\, \varphi(T\alpha'') \\
&\qquad = \textstyle\sum \alpha'' \in D'' \,.\, \varphi(T\alpha''))\ .
\end{aligned}
$$

If we view the above proof in the light of positive summation by partition we become convinced of

Theorem

7.37 ($\varphi \in \mathrm{gauge} \to \mathrm{grid}\, \varphi = \mathrm{scheme}\, \varphi$)

We also have

Theorem

7.38 ($D \in \mathrm{scheme}\, \varphi \to D \in \mathrm{grid}\, \varphi \leftrightarrow \bigwedge T \in \mathrm{dmn}\, \varphi \bigwedge S \in \mathrm{scheme}\, \varphi(\sum \beta \in D \sqcap S \,.\, \varphi(T\beta) \in \mathrm{spl})$)

Lemma

7.39 ($D \in \mathrm{sb} \sim 1 \wedge \mathrm{swing}\, fD\varphi = N < \infty \to \mathrm{osum}\, fD\varphi \leq 2 \cdot N$)

Proof:

We notice first that

$(N \geq 0)$

and

$(\xi \in \mathrm{swath}\, D \wedge \eta \in \mathrm{swath}\, D \to N \leq \sum \beta \in D\, \mathrm{prt}'((.\, f \,.\, \xi\beta \;—\; .\, f \,.\, \eta\beta) \bullet .\, \varphi\beta))$.

Next we check that

$$
\begin{aligned}
&(\xi \in \mathrm{swath}\, D \wedge \eta \in \mathrm{swath}\, D \wedge \\
&\quad \textstyle\bigwedge \beta(\underline{\mathrm{u}}\beta = \sqsubseteq t(\mathrm{prt}'((.\, f \,.\, \xi\beta \;—\; .\, f \,.\, \eta\beta) \bullet .\, \varphi\beta) \geq 0)) \wedge \\
&\quad \xi' = \curlywedge \beta \in D(\underline{\mathrm{u}}\beta \cap .\, \xi\beta \cup \sim\underline{\mathrm{u}}\beta \cap .\, \eta\beta) \wedge \\
&\quad \eta' = \curlywedge \beta \in D(\underline{\mathrm{u}}B \cap .\, \eta\beta \cup \sim\underline{\mathrm{u}}\beta \cap .\, \xi\beta) \wedge \\
&\quad N \geq \textstyle\sum \beta \in D\, \mathrm{prt}'((.\, f \,.\, \xi'\beta \;—\; .\, f \,.\, \eta'\beta) \bullet .\, \varphi\beta) \\
&\qquad = \textstyle\sum \beta \in D |\, \mathrm{prt}'((.\, f \,.\, \xi\beta \;—\; .\, f\eta\beta) \bullet .\, \varphi\beta)| \\
&\quad \to N \geq \textstyle\sum \beta \in D |\, \mathrm{prt}'((.\, f \,.\, \xi\beta \;—\; .\, f \,.\, \eta\beta) \bullet .\, \varphi\beta)|)\ .
\end{aligned}
$$

In the same way check that

$(\xi \in \mathrm{swath}\, D \wedge \eta \in \mathrm{swath}\, D \to N \geq \sum \beta \in D |\, \mathrm{prt}''((.\, f \,.\, \xi\beta \;—\; .\, f \,.\, \eta\beta) \bullet .\, \varphi\beta)|)$.

We now know

$(\xi \in \operatorname{swath} D \wedge \eta \in \operatorname{swath} D \to 2 \cdot N \geq \sum \beta \in D|(\mathbf{.}\, f\, \mathbf{.}\, \xi\beta \,—\, \mathbf{.}\, f\, \mathbf{.}\, \eta\beta) \bullet \mathbf{.}\, \varphi\beta|)$.

From this we learn

$(\beta \in D \wedge x \in \beta \wedge y \in \beta \to |(\mathbf{.}\, fx \,—\, \mathbf{.}\, fy) \bullet \mathbf{.}\, \varphi\beta| \leq 2 \cdot N)$

and hence

$(\beta \in D \to |\operatorname{oscl} f\beta \bullet \mathbf{.}\, \varphi\beta| \leq 2 \cdot N < \infty)$.

Accordingly we conclude

$(1 < \lambda < \infty \to \bigvee \xi, \eta, \in \operatorname{swath} D($
$\bigwedge \beta \in D(|\operatorname{oscl} f\beta \bullet \mathbf{.}\, \varphi\beta| \leq \lambda \cdot |(\mathbf{.}\, f\, \mathbf{.}\, \xi\beta \,—\, \mathbf{.}\, f\, \mathbf{.}\, \eta\beta) \bullet \mathbf{.}\, \varphi\beta|) \wedge$
$\operatorname{osum} fD\varphi$
$= \sum \beta \in D|\operatorname{oscl} fD\beta \bullet \mathbf{.}\, \varphi\beta|$
$\leq \sum \beta \in D(\lambda \cdot |(\mathbf{.}\, f\, \mathbf{.}\, \xi\beta \,—\, \mathbf{.}\, f\, \mathbf{.}\, \eta\beta) \bullet \mathbf{.}\, \varphi\beta|)$
$= \lambda \cdot \sum \beta \in D|(\mathbf{.}\, f\, \mathbf{.}\, \xi\beta \,—\, \mathbf{.}\, f\, \mathbf{.}\, \eta\beta) \bullet \mathbf{.}\, \varphi\beta|$
$\leq \lambda \cdot 2 \cdot N))$

and

$(1 < \lambda < \infty \to \operatorname{osum} fD\varphi \leq \lambda \cdot 2 \cdot N)$

and

$(\operatorname{osum} fD\varphi \leq 2 \cdot N)$.

Lemma

7.41 $(\operatorname{osum} fD\varphi < \infty \to \operatorname{swing} fD\varphi \leq \operatorname{osum} fD\varphi)$

Proof:

$(\xi \in \operatorname{swath} D \wedge \eta \in \operatorname{swath} D \to$
$\infty \geq |\operatorname{rsum} f\xi\varphi \,—\, \operatorname{rsum} f\eta\varphi|$
$= |\sum \beta \in D((\mathbf{.}\, f\, \mathbf{.}\, \xi\beta \,—\, \mathbf{.}\, f\, \mathbf{.}\, \eta\beta) \bullet \mathbf{.}\, \varphi\beta)|$
$\leq \sum \beta \in D|(\mathbf{.}\, f\, \mathbf{.}\, \xi\beta \,—\, \mathbf{.}\, f\, \mathbf{.}\, \eta\beta) \bullet \mathbf{.}\, \varphi\beta|$
$\leq \sum \beta \in D|\operatorname{oscl} f\beta \bullet \mathbf{.}\, \varphi\beta|$
$= \operatorname{osum} fD\varphi)$

From 7.41 and 7.39 we infer

Theorem

7.42 $(D \in \operatorname{sb} \sim 1 \to 0 \leq \operatorname{swing} fD\varphi \leq \operatorname{osum} fD\varphi \leq 2 \cdot \operatorname{swing} fD\varphi \leq \infty)$

Theorem

7.43 $(\int \underline{\mathrm{u}} x\varphi \, \mathrm{d}\, x \in \mathrm{kf} \to \overline{\underline{\int}} \underline{\mathrm{u}} x\varphi \, \mathrm{d}\, x = 0)$

7.44 Lemmas

.0 $(\varphi \in \operatorname{gauge} \wedge D \in \operatorname{grid} \varphi \to \sum \beta \in D|\mathbf{.}\, \varphi(T\beta)| \leq |\mathbf{.}\, \varphi T|)$

.1 $(\mathbf{.}\, \psi 0 = 0 \wedge D \subset\subset F \in \operatorname{dsjn} \to \sum \beta \in D(\underline{\mathrm{u}}\beta \bullet \mathbf{.}\, \psi\beta) = \sum \beta \in D \sum \gamma \in F(\underline{\mathrm{u}}\beta \bullet \mathbf{.}\, \psi(\beta\gamma)))$

.2 $(\varphi \in \operatorname{gauge} \wedge D \in \operatorname{grid} \varphi \wedge G \in \operatorname{grid} \varphi \wedge S = \sum \alpha \in D(\underline{\mathrm{u}}\alpha \bullet \mathbf{.}\, \varphi\alpha) \in \mathrm{spl}$
$\to \sum \alpha \in D \sum \beta \in G(\underline{\mathrm{u}}\alpha \bullet \mathbf{.}\, \varphi(\alpha\beta)) = S = \sum \beta \in G \sum \alpha \in D(\underline{\mathrm{u}}\alpha \bullet \mathbf{.}\, \varphi(\alpha\beta)))$

.3 $(\varphi \in \operatorname{gauge} \wedge M = \operatorname{mode} \varphi \wedge D \in \operatorname{grid} \varphi \wedge \operatorname{osum} fD\varphi < \infty \wedge \eta \in \operatorname{vs} MD \to \operatorname{rsum} f\eta\varphi \in \mathrm{kf})$

Proof:

According to 7.40 and .2

$(\operatorname{rsum} f\eta\varphi = \sum \beta \in \operatorname{dmn} \eta \sum \gamma \in D(\mathbf{.}\, f\, \mathbf{.}\, \eta\beta \bullet \mathbf{.}\, \varphi\gamma)))$.

Evidently

$(\beta \in \operatorname{dmn}\eta \wedge \gamma \in D \rightarrow |(\mathbf{.}\, f\, \mathbf{.}\, \eta\beta - \mathbf{.}\, f\, \mathbf{.}\, \lambda\gamma) \bullet \mathbf{.}\, \varphi(\beta\gamma)| \le |\operatorname{oscl} f\gamma \bullet \mathbf{.}\, \varphi(\beta\gamma)|)$.

Now because of this and .0

$$
\begin{aligned}
(\lambda \in \operatorname{swath} D \rightarrow \\
&\textstyle\sum \beta \in \operatorname{dmn}\eta \sum \gamma \in D|(\mathbf{.}\, f\, \mathbf{.}\, \eta\beta - \mathbf{.}\, f\, \mathbf{.}\, \lambda\gamma) \bullet \mathbf{.}\, \varphi(\beta\gamma)| \\
&\le \textstyle\sum \beta \in \operatorname{dmn}\eta \sum \gamma \in D|\operatorname{oscl} f\gamma \bullet \mathbf{.}\, \varphi(\beta\gamma)| \\
&= \textstyle\sum \gamma \in D \sum \beta \in \operatorname{dmn}\eta(\operatorname{oscl} f\gamma \bullet |\mathbf{.}\, \varphi(\beta\gamma)|) \\
&= \textstyle\sum \gamma \in D(\operatorname{oscl} f\gamma \bullet \sum \beta \in \operatorname{dmn}\eta|\mathbf{.}\, \varphi(\beta\gamma)|) \\
&\le \textstyle\sum \gamma \in D(\operatorname{oscl} f\gamma \bullet |\mathbf{.}\, \varphi\gamma|) \\
&= \operatorname{osum} fD\varphi \\
&< \infty) .
\end{aligned}
$$

Because of 7.40 and the above

$(\operatorname{rsum} f\eta\varphi \in \mathrm{kf})$.

.4 $(\varphi \in \operatorname{gauge} \wedge M = \operatorname{mode}\varphi \wedge D \in \operatorname{grid}\varphi \wedge \operatorname{osum} fD\varphi < \infty \wedge$
$\xi \in \operatorname{vs} MD \wedge \eta \in \operatorname{vs} MD \wedge F = \operatorname{dmn}\xi \wedge G = \operatorname{dmn}\eta$
$\rightarrow \operatorname{rsum} f\xi\varphi - \operatorname{rsum} f\eta\varphi = \sum \alpha \in F \sum \beta \in G((\mathbf{.}\, f\, \mathbf{.}\, \xi\alpha - \mathbf{.}\, f\, \mathbf{.}\, \eta\beta) \bullet \mathbf{.}\, \varphi(\alpha\beta)) \wedge$
$\sum \alpha \in F \sum \beta \in G|(\mathbf{.}\, f\, \mathbf{.}\, \xi\alpha - \mathbf{.}\, f\, \mathbf{.}\, \eta\beta) \bullet \mathbf{.}\, \varphi(\alpha\beta)| \le \operatorname{osum} fD\varphi)$

Proof:

Because of .3 and .2

$$
\begin{aligned}
(\mathrm{kf} \ni \operatorname{rsum} f\xi\varphi - \operatorname{rsum} f\eta\varphi \\
&= \textstyle\sum \alpha \in F \sum \beta \in G(\mathbf{.}\, f\, \mathbf{.}\, \xi\alpha \bullet \mathbf{.}\, \varphi(\alpha\beta)) - \sum \alpha \in F \sum \beta \in G(\mathbf{.}\, f\, \mathbf{.}\, \eta\beta \bullet \mathbf{.}\, \varphi(\alpha\beta)) \\
&= \textstyle\sum \alpha \in F \sum \beta \in G((\mathbf{.}\, f\, \mathbf{.}\, \xi\alpha - \mathbf{.}\, f\, \mathbf{.}\, \eta\beta) \bullet \mathbf{.}\, \varphi(\alpha\beta))) .
\end{aligned}
$$

Evidently

$$
\begin{aligned}
(\alpha \in F \wedge \beta \in G \wedge \gamma \in D \rightarrow \\
&|\mathbf{.}\, f\, \mathbf{.}\, \xi\alpha - \mathbf{.}\, f\, \mathbf{.}\, \eta\beta| \bullet |\mathbf{.}\, \varphi(\alpha\beta\gamma)| \le \operatorname{oscl} f\gamma|\mathbf{.}\, \varphi(\alpha\beta\gamma)|) .
\end{aligned}
$$

Because of this, .1, and .0,

$$
\begin{aligned}
(\textstyle\sum \alpha \in F \sum \beta \in G|(\mathbf{.}\, f\, \mathbf{.}\, \xi\alpha - \mathbf{.}\, f\, \mathbf{.}\, \eta\beta) \bullet \mathbf{.}\, \varphi(\alpha\beta)| \\
&= \textstyle\sum \alpha \in F \sum \beta \in G(|\mathbf{.}\, f\, \mathbf{.}\, \xi\alpha - \mathbf{.}\, f\, \mathbf{.}\, \eta\beta| \bullet |\mathbf{.}\, \varphi(\alpha\beta)|) \\
&= \textstyle\sum \alpha \in F \sum \beta \in G \sum \gamma \in D(|\mathbf{.}\, f\, \mathbf{.}\, \xi\alpha - \mathbf{.}\, f\, \mathbf{.}\, \eta\beta| \bullet |\mathbf{.}\, \varphi(\alpha\beta\gamma)|) \\
&\le \textstyle\sum \alpha \in F \sum \beta \in G \sum \gamma \in D(\operatorname{oscl} f\gamma \bullet |\mathbf{.}\, \varphi(\alpha\beta\gamma)|) \\
&= \textstyle\sum \gamma \in D \sum \alpha \in F \sum \beta \in G(\operatorname{oscl} f\gamma \bullet |\mathbf{.}\, \varphi(\alpha\beta\gamma)|) \\
&= \textstyle\sum \gamma \in D(\operatorname{oscl} f\gamma \bullet \sum \alpha \in F \sum \beta \in G|\mathbf{.}\, \varphi(\alpha\beta\gamma)|) \\
&\le \textstyle\sum \gamma \in D(\operatorname{oscl} f\gamma \bullet \sum \alpha \in F|\mathbf{.}\, \varphi(\alpha\gamma)|) \\
&\le \textstyle\sum \gamma \in D(\operatorname{oscl} f\gamma \bullet |\mathbf{.}\, \varphi\gamma|) \\
&= \operatorname{osum} fD\varphi) .
\end{aligned}
$$

The proof is complete

From .4 we easily infer

Theorem

7.45 $(\varphi \in \operatorname{gauge} \wedge M = \operatorname{mode}\varphi \wedge D \in \operatorname{grid}\varphi \wedge \xi \in \operatorname{vs} MD \wedge \eta \in \operatorname{vs} MD$
$\rightarrow |\operatorname{rsum} f\xi\varphi - \operatorname{rsum} f\eta\varphi| \le \operatorname{osum} fD\varphi)$

It is not hard to check the general

Theorem

7.46 $(\operatorname{run}\operatorname{is} R \wedge \operatorname{space}\rho \in \operatorname{Complete}\rho \wedge \bigwedge \epsilon > 0 \bigvee \delta \in \operatorname{dmn} R \bigwedge x \in \operatorname{vs} R\delta \bigwedge y \in \operatorname{vs} R\delta(\mathbf{.}\, \rho(\mathbf{.}\, fx, \mathbf{.}\, fy) \le \epsilon)$
$\rightarrow \bigvee P(\operatorname{lm} xR\, \mathbf{.}\, \rho(\mathbf{.}\, fx, P) = 0))$

Since (kf $\in$ Complete $\underline{\text{kf}}$) we infer from 7.46, 7.45, and 7.43 the beautiful
Theorem
7.47 $(\varphi \in \text{gauge} \to \int \underline{\text{u}}x\varphi \, \mathrm{d}\, x \in \text{kf} \leftrightarrow \overline{\underline{\int}} \underline{\text{u}}x\varphi \, \mathrm{d}\, x = 0)$

7.48 Theorems

.0 $(\varphi \in \text{grator} \to \overline{\underline{\int}} \underline{\text{u}}x\varphi \, \mathrm{d}\, x \geq \overline{\underline{\int}} |\underline{\text{u}}x|\varphi \, \mathrm{d}\, x)$

.1 $(\varphi \in \text{gauge} \wedge \int \underline{\text{u}}x\varphi \, \mathrm{d}\, x \in \text{kf} \to 0 \leq \int |\underline{\text{u}}x|\varphi \, \mathrm{d}\, x < \infty)$

.2 $(\varphi \in \text{gauge} \wedge \text{alm}\, \varphi x(\underline{\text{u}}x \in \text{rl}) \wedge \int \underline{\text{u}}x\varphi \, \mathrm{d}\, x \in \text{rf} \to \int \underline{\text{u}}x\varphi \, \mathrm{d}\, x = \int \text{ps}\, \underline{\text{u}}x\varphi \, \mathrm{d}\, x — \int \text{ng}\, \underline{\text{u}}x\varphi \, \mathrm{d}\, x)$

.3 $(\varphi \in \text{gauge} \wedge \int \underline{\text{u}}x\varphi \, \mathrm{d}\, x \in \text{kf} \to |\int \underline{\text{u}}x\varphi \, \mathrm{d}\, x| \leq \int |\underline{\text{u}}x|\varphi \, \mathrm{d}\, x < \infty)$

Proof:
So choose c that
$(|c| = 1 \wedge c \cdot \int \underline{\text{u}}x\varphi \, \mathrm{d}\, x = |\int \underline{\text{u}}x\varphi \, \mathrm{d}\, x|)$.
Because of .1 and 7.15 and 7.16

$$
\begin{aligned}
(|\textstyle\int \underline{\text{u}}x\varphi \, \mathrm{d}\, x| \\
&= c \cdot \textstyle\int \underline{\text{u}}x\varphi \, \mathrm{d}\, x \\
&= \textstyle\int (c \cdot \underline{\text{u}}x)\varphi \, \mathrm{d}\, x \\
&= \text{prt}' \textstyle\int (c \cdot \underline{\text{u}}x)\varphi \, \mathrm{d}\, x \\
&= \textstyle\int \text{prt}'(c \cdot \underline{\text{u}}x)\varphi \, \mathrm{d}\, x \\
&\leq \textstyle\int |c \cdot \underline{\text{u}}x|\varphi \, \mathrm{d}\, x \\
&= \textstyle\int |\underline{\text{u}}x|\varphi \, \mathrm{d}\, x) \\
&< \infty) \, .
\end{aligned}
$$

It is not difficult to check
Theorems

7.49 $(\varphi \in \text{grator} \wedge \theta = \text{vrn}\, \varphi \to$

.0 $\theta \in \text{grator} \cap \text{gauge} \cap \text{On}\, \text{dmn}\, \varphi \wedge$

.1 $\text{grid}\, \varphi \subset \text{grid}\, \theta \wedge$

.2 $\bigwedge T \in \text{dmn}\, \varphi(|\,\boldsymbol{.}\,\varphi T| \leq \boldsymbol{.}\,\theta T))$

Dominated summation by distribution helps us check
Lemma
7.50 $(\theta = \text{vrn}\, \varphi \wedge D \in \text{grid}\, \varphi \wedge G \in \text{grid}\, \varphi \wedge \sum \alpha \in D(\underline{\text{u}}\alpha \bullet \boldsymbol{.}\,\theta\alpha) \in \text{kf} \wedge S = \sum \alpha \in D(\underline{\text{u}}\alpha \bullet \boldsymbol{.}\,\varphi\alpha)$
$\to S \in \text{kf} \wedge \sum \alpha \in D \sum \beta \in G(\underline{\text{u}}\alpha \bullet \boldsymbol{.}\,\varphi(\alpha\beta)) = S = \sum \beta \in G \sum \alpha \in D(\underline{\text{u}}\alpha \bullet \boldsymbol{.}\,\varphi(\alpha\beta)))$

Lemma
7.51 $(\theta = \text{vrn}\, \varphi \wedge M = \text{mode}\, \varphi \wedge D \in \text{grid}\, \varphi \wedge \text{osum}\, fD\varphi < \infty$
$\wedge\, \xi \in \text{vs}\, MD \wedge \eta \in \text{vs}\, MD \wedge F = \text{dmn}\, \xi \wedge G = \text{dmn}\, \eta$
$\to |\,\text{rsum}\, f\xi\varphi — \text{rsum}\, f\eta\varphi| \leq \text{osum}\, fD\theta)$
Proof:
Because of 7.50

$$
\begin{aligned}
(\text{kf} \ni \text{rsum}\, f\xi\varphi — \text{rsum}\, f\eta\varphi \\
&= \textstyle\sum \alpha \in F \sum \beta \in G(\boldsymbol{.}\, f \,\boldsymbol{.}\, \xi\alpha \bullet \boldsymbol{.}\,\varphi(\alpha\beta)) — \sum \alpha \in F \sum \beta \in G(\boldsymbol{.}\, f \,\boldsymbol{.}\, \eta\beta \bullet \boldsymbol{.}\,\varphi(\alpha\beta)) \\
&= \textstyle\sum \alpha \in G \sum \beta \in G((\boldsymbol{.}\, f \,\boldsymbol{.}\, \xi\alpha — \boldsymbol{.}\, f \,\boldsymbol{.}\, \eta\beta) \bullet \boldsymbol{.}\,\varphi(\alpha\beta))) \, .
\end{aligned}
$$

Because of this and 7.44.4

$(|\operatorname{rsum} f\xi\varphi - \operatorname{rsum} f\eta\varphi|$
$= |\sum \alpha \in F \sum \beta \in G((\mathbf{.}\, f \,\mathbf{.}\, \xi\alpha - \mathbf{.}\, f \,\mathbf{.}\, \eta\beta) \bullet \mathbf{.}\, \varphi(\alpha\beta))|$
$\leq \sum \alpha \in F \sum \beta \in G|(\mathbf{.}\, f \,\mathbf{.}\, \xi\alpha - \mathbf{.}\, f \,\mathbf{.}\, \eta\beta) \bullet \mathbf{.}\, \varphi(\alpha\beta)|$
$\leq \sum \alpha \in F \sum \beta \in G|(\mathbf{.}\, f \,\mathbf{.}\, \xi\alpha - \mathbf{.}\, f \,\mathbf{.}\, \eta\beta) \bullet \mathbf{.}\, \theta(\alpha\beta)|$
$= \operatorname{osum} f D\theta)$.

The proof is complete.

From 7.51 we easily infer

Theorem

7.52 $(\theta = \operatorname{vrn}\varphi \wedge M = \operatorname{mode}\varphi \wedge D \in \operatorname{grid}\varphi \wedge \xi \in \operatorname{vs} MD \wedge \eta \in \operatorname{vs} MD$
$\rightarrow |\operatorname{rsum} f\xi\varphi - \operatorname{rsum} f\eta\varphi| \leq \operatorname{osum} f D\theta)$

Theorem

7.53 $(\theta = \operatorname{vrn}\varphi \wedge M = \operatorname{mode}\varphi \rightarrow$
.0 $(\int\# M\underline{\mathrm{u}}x\theta \,\mathrm{d}\,x \in \mathrm{kf} \rightarrow \int \underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x \in \mathrm{kf}) \wedge$
.1 $(\varphi \in \operatorname{gridor} \rightarrow \operatorname{grid}\varphi = \operatorname{grid}\theta) \wedge$
.2 $(\varphi \in \operatorname{gridor} \rightarrow M = \operatorname{mode}\theta) \wedge$
.3 $(\varphi \in \operatorname{gridor} \wedge \int \underline{\mathrm{u}}x\theta \,\mathrm{d}\,x \in \mathrm{kf} \rightarrow \int \underline{\mathrm{u}}x\varphi \,\mathrm{d}\,x \in \mathrm{kf}))$

Integration of Step Functions

Definition
7.54 $(\text{step}\,\lambda \equiv \measuredangle x \in \text{rlm}\,\lambda \sum \beta \in \text{dmn}\,\lambda(\mathbf{.}\,\lambda\beta \bullet \text{Cr}\,x\beta))$

Theorem
7.55 $(x \in \beta \in \text{dmn}\,\lambda \in \text{dsjn} \wedge \mathbf{.}\,\lambda\beta \in \text{spl} \wedge f = \text{step}\,\lambda \rightarrow \mathbf{.}\,fx = \mathbf{.}\,\lambda\beta)$

Lemma
7.56 $(\varphi \in \text{gauge} \wedge M = \text{mode}\,\varphi \wedge D \in \text{grid}\,\varphi \wedge \lambda \in \text{On}\,D \wedge f = \text{step}\,\lambda \wedge$
$S = \sum \gamma \in D(\mathbf{.}\,\lambda\gamma \bullet \mathbf{.}\,\varphi\gamma) \in \text{spl} \wedge \xi \in \text{vs}\,MD$
$\rightarrow \text{rsum}\,f\xi\varphi = S)$
Proof:
Let
$(F = \text{dmn}\,\xi)$.
Note first that
.0 $(\beta \in F \wedge \gamma \in D \rightarrow \mathbf{.}\,f\,\mathbf{.}\,\xi\beta \bullet \mathbf{.}\,\varphi(\beta\gamma) = \mathbf{.}\,\lambda\gamma \bullet \mathbf{.}\,\varphi(\beta\gamma))$
and then use 7.44.2, .0, and 7.44.1 in concluding
$(S = \sum \beta \in F \sum \gamma \in D(\mathbf{.}\,\lambda\gamma \bullet \mathbf{.}\,\varphi(\beta\gamma))$
$= \sum \beta \in F \sum \gamma \in D(\mathbf{.}\,f\,\mathbf{.}\,\xi\beta \bullet \mathbf{.}\,\varphi(\beta\gamma))$
$= \sum \beta \in F(\mathbf{.}\,f\,\mathbf{.}\,\xi\beta \bullet \mathbf{.}\,\varphi\beta)$
$= \text{rsum}\,f\xi\varphi)$.

We now have at once
Theorem
7.57 $(\varphi \in \text{gauge} \wedge D \in \text{grid}\,\varphi \wedge \lambda \in \text{On}\,D \wedge f = \text{step}\,\lambda \wedge S = \sum \beta \in D(\mathbf{.}\,\lambda\beta \bullet \mathbf{.}\,\varphi\beta) \in \text{spl} \rightarrow \int f\varphi\} = S)$

We also have
Theorem
7.58 $(\varphi \in \text{gauge} \wedge D \in \text{grid}\,\varphi \wedge \lambda \in \text{On}\,D \wedge f = \text{step}\,\lambda \wedge S = \sum \beta \in D(\mathbf{.}\,\lambda\beta \bullet \mathbf{.}\,\varphi\beta) \in \text{spl} \wedge$
$\bigwedge \beta \in D(\mathbf{.}\,\lambda\beta \bullet \mathbf{.}\,\varphi\beta \in \text{kf}) \rightarrow \int \mathbf{.}\,fx\varphi\,\mathrm{d}\,x = S)$

Lemma
7.59 $(M = \text{mode}\,\varphi \wedge A \in D \in \text{grid}\,\varphi \wedge A \in \text{dmn}\,\varphi \wedge f = \measuredangle x\,\text{Cr}\,xA \wedge \xi \in \text{vs}\,MD \rightarrow \text{rsum}\,f\xi\varphi = \mathbf{.}\,\varphi A)$
Proof:
Simply note
$(\beta \in \text{dmn}\,\xi \rightarrow \mathbf{.}\,f\,\mathbf{.}\,\xi\beta = \mathbf{.}\,\varphi(\beta A))$.
and apply 7.19.

We now have at once
Theorem
7.60 $(A \in D \in \text{grid}\,\varphi \wedge A \in \text{dmn}\,\varphi \wedge f = \measuredangle x\,\text{Cr}\,xA \rightarrow \int f\varphi\} = \mathbf{.}\,\varphi A)$

We also have
7.61 $(A \in D \in \text{grid}\,\varphi \wedge A \in \text{dmn}'\,\varphi \rightarrow \int A; 1\varphi\,\mathrm{d}\,x = \mathbf{.}\,\varphi A)$

More general is
Theorem
7.62 $(A \in D \in \operatorname{grid} \varphi \wedge A \in \operatorname{dmn} \varphi \wedge G \in \operatorname{grid} \varphi \wedge \bigwedge \beta \in G(\beta A \in \operatorname{dmn}' \varphi) \rightarrow \int A; 1\varphi \, \mathrm{d}\, x = \mathbf{.}\, \varphi A)$

In keeping with 7.57 we have
Theorem
7.63 $(D \in \operatorname{grid} \varphi \wedge \lambda \in \operatorname{On} D \wedge f = \operatorname{step} \lambda \wedge \int f\varphi\} \in \operatorname{spl} \wedge S = \sum \beta \in D(\mathbf{.}\, \lambda\beta \bullet \mathbf{.}\, \varphi\beta) \in \operatorname{spl} \rightarrow \int f\varphi\} = S)$

In keeping wtih 7.58 we have
Theorem
7.64 $(D \in \operatorname{grid} \varphi \wedge \lambda \in \operatorname{On} D \wedge f = \operatorname{step} \lambda \wedge \int f\varphi\} \in \operatorname{spl} \wedge S = \sum \beta \in D(\mathbf{.}\, \lambda\beta \bullet \mathbf{.}\, \varphi\beta) \in \operatorname{spl} \wedge$
$\bigwedge \beta \in D(\mathbf{.}\, \lambda\beta \bullet \mathbf{.}\, \varphi\beta \in \operatorname{kf}) \rightarrow \int \mathbf{.}\, fx\varphi \, \mathrm{d}\, x = S)$

Completely Additive Functions

7.65 Definitions

.0 (additive$''$ $\varphi \equiv$ E $A \in \operatorname{dmn} \varphi \wedge$ function is $\varphi(\operatorname{dmn}\varphi \cap \operatorname{sb} A \in \nabla\text{field} \wedge$
$\bigwedge G \in \text{cbl} \cap \text{dsjn} \cap \text{sb}\operatorname{dmn}\varphi(.\,\varphi(A \,\nabla\, G) = \sum \beta \in G\,.\,\varphi(A\beta) \in \text{spl})))$

.1 (addor $\equiv$ E $\varphi \in$ grator(dmn $\varphi \in$ additive$''$ φ))

.2 (addor$''$ $\equiv$ E $\varphi \in$ addor $\bigvee S \in$ sqnc$\subset$ dmn φ(rlm $\varphi = \bigvee n \in \omega\,.\,Sn$))

.3 (dor+ $\varphi \equiv \curlywedge \beta \in$ dmn φ sup $\alpha \in$ dmn $\varphi \cap$ sb $\beta\,.\,\varphi\alpha$)

.4 (dor — $\varphi \equiv$ dor+ $-\varphi$)

.5 (dor$\pm$ $\varphi \equiv$ (dor+ φ — dor — φ))

.6 (mr $\varphi \equiv$ mss φ rlm φ dmn φ)

.7 (var $\varphi \equiv$ mr vrn φ)

.8 (dor$'$ $\varphi \equiv \curlywedge \beta$ prt$'\,.\,\varphi\beta$)

.9 (dor$''$ $\varphi \equiv \curlywedge \beta$ prt$''\,.\,\varphi\beta$)

Our interest in 7.65.9 and in 7.66 below arises from the behavior of the imaginary part of a function in addor.

Definition

7.65A (∇meet $\equiv$ E $F \in \sim 1 \bigwedge A \in F(\bigvee \beta \in F$ sng$(A\beta) = F$ sb $A \in \nabla$field))

7.66 Definitions

.0 (dmn+ $\varphi \equiv$ E $\beta(.\,\beta > 0)$)

.1 (paddor $\equiv$ E $\varphi \in$ grator $\cap$ To rl(dmn $\varphi \in \nabla$meet $\wedge$ dmn+ $\varphi \subset$ additive$''$ $\wedge$
$\bigwedge A \in$ dmn $\varphi \bigwedge \alpha, \beta, \in$ dmn+ $\varphi \cap$ sb $A(\alpha\beta = 0 \to \alpha \cup \beta \in$ dmn+ φ)))

.2 (topaddor $\equiv$ E $\varphi \in$ paddor $\bigwedge G \in$ cbl $\cap$ dsjn $\cap$ sb dmn $\varphi \bigwedge A \in$ dmn $\varphi(\sum \beta \in G\,.\,\varphi(A\beta) < \infty$))

7.67 Theorems

.0 ($T \in$ additive$''$ $\varphi \wedge A \in$ dmn $\varphi \to .\,\varphi T = .\,\varphi(TA) + .\,\varphi(T{\sim}A)$)

.1 ($T \in$ additive$''$ $\varphi \wedge A \in$ dmn $\varphi \wedge B \in$ dmn $\varphi \to .\,\varphi(TA) + .\,\varphi(TB) = .\,\varphi(TA \cup TB) + .\,\varphi(TAB)$)

7.68 Theorems

.0 ($c \in$ rf $\sim 1 \wedge \varphi \in$ addor $\to c \cdot \varphi \in$ addor On dmn φ)

.1 ($c \in$ kf $\wedge\ \varphi \in$ addor $\cap$ To kf $\to c \cdot \varphi \in$ addor On dmn φ To kf)

Perhaps not altogether expected is

Theorem

7.69 ($\varphi_1 \in$ addor $\wedge\ \varphi_2 \in$ addor $\wedge\ \varphi = \varphi_1 + \varphi_2 \to$

.0 $\bigwedge A \in$ dmn φ(dmn $\varphi \cap$ sb $A =$ dmn $\varphi_1 \cap$ dmn $\varphi_2 \cap$ sb A) $\wedge$

.1 $\varphi \in$ addor)

7.70 Theorems

.0 ($A \in$ additive$''$ $\varphi \to$ strc φ sb $A \in$ addor)

.1 ($\varphi \in$ addor $\to$ strc φ dmn$'$ $\varphi \in$ addor)

7.71 Lemmas

.0 $(\varphi \in \mathrm{grator} \cap \mathrm{To}\,\mathrm{rl} \rightarrow \mathrm{dor}{+}\,\varphi \in \mathrm{gauge}\,\mathrm{On}\,\mathrm{dmn}\,\varphi)$

.1 $(\varphi \in \mathrm{paddor} \wedge \theta = \mathrm{dor}{\pm}\,\varphi \wedge A \in \mathrm{dmn}\,\theta \wedge G \in \mathrm{cbl}\,\mathrm{dsjn}\,\mathrm{sb}\,\mathrm{dmn}\,\varphi$

$\rightarrow 0 \leq .\,\theta(A \triangledown G) = \sum \beta \in G\,.\,\theta(\beta A))$

Proof:

On the one hand

$(.\,\theta(A \triangledown G) > \sum \beta \in G\,.\,\theta(\beta A) \rightarrow \bigvee \alpha \in \mathrm{dmn}\,\varphi \cap \mathrm{sb}(A \triangledown G)$

$(.\,\varphi\alpha$

$> \sum \beta \in G\,.\,\theta(\beta A)$

$\geq \sum \beta \in G\,.\,\theta(\beta\alpha)$

$\geq 0 \wedge$

$.\,\varphi\alpha$

$> \sum \beta \in G\,.\,\varphi(\beta\alpha)$

$= .\,\varphi(\alpha \triangledown G)$

$= .\,\varphi\alpha \wedge$

$.\,\varphi\alpha > .\,\varphi\alpha))$.

On the other hand

$(.\,\theta(A \triangledown G) < \sum \beta \in G\,.\,\theta(\beta A) \rightarrow \bigvee H \in \mathrm{fnt} \cap \mathrm{sb}\,G($

$.\,\theta(A \triangledown H) < \sum \beta \in H\,.\,\theta(\beta A) \wedge$

$H \neq 0 \wedge$

$\bigwedge \beta \in H(.\,\theta(\beta A) > 0) \wedge$

$\bigvee \xi(\bigwedge \beta \in H(.\,\xi\beta \in \mathrm{dmn}{+}\,\varphi \cap \mathrm{sb}(\beta A)) \wedge$

$\sum \beta \in H\,.\,\varphi\,.\,\xi\beta$

$> .\,\theta(A \triangledown H)$

$\geq .\,\theta(A \bigvee \beta \in H\,.\,\xi\beta)$

$= .\,\theta \bigvee \beta \in H\,.\,\xi\beta$

$\geq .\,\varphi \bigvee \beta \in H\,.\,\xi\beta$

$= \sum \beta \in H\,.\,\varphi\,.\,\xi\beta \wedge$

$\sum \beta \in H\,.\,\varphi\,.\,\xi\beta > \sum \beta \in H\,.\,\varphi\,.\,\xi\beta)))$.

The desired conclusion follows.

Helped by 7.66 and 7.71 we infer

7.72 Theorems

.0 $(\mathrm{addor}\,\mathrm{To}\,\mathrm{rl} \subset \mathrm{paddor})$

.1 $(\varphi \in \mathrm{paddor} \wedge \theta = \mathrm{dor}{+}\,\varphi \rightarrow \theta \in \mathrm{addor}\,\mathrm{gauge}\,\mathrm{On}\,\mathrm{dmn}\,\varphi)$

Theorem

7.73 $(A \in \mathrm{additive}''\,\varphi\,\mathrm{dmn}'\,\varphi \rightarrow \sup \alpha \in \mathrm{dmn}\,\varphi \cap \mathrm{sb}\,A |\,.\,\varphi\alpha| < \infty)$

Proof:

Note that

$(\mathrm{dmn}\,\varphi \cap \mathrm{sb}\,A \subset \mathrm{dmn}'\,\varphi)$

and let

$(F = \mathrm{dmn}\,\varphi \cap \mathrm{sb}\,A \wedge N = \mathsf{E}\,\beta \in F(\sup \alpha \in F \cap \mathrm{sb}\,\beta |\,.\,\varphi\alpha| = \infty) \wedge M = \mathsf{E}\,\beta \in F(|\,.\,\varphi\beta| \geq 1))$.

After checking

$(C \in N \rightarrow \bigvee D \subset C(D \in M \wedge C{\sim}D \in MN))$

we readily infer

$(B \in F \wedge A{\sim}B \in N \rightarrow \bigvee B' \supset B(B'{\sim}B \in M \wedge A{\sim}B' \in N))$.

Because of this we know

$(A \in N$

$\rightarrow \bigvee S \in \text{sqnc}{\subset} F($

$\textbf{.}S0 = 0 \wedge$

$\bigwedge n \in \omega(\textbf{.}S(n+1){\sim}\textbf{.}Sn \in M \wedge A{\sim}\textbf{.}S(n+1) \in N) \wedge$

∞

$> \sum n \in \omega |\textbf{.}\varphi(\textbf{.}S(n+1){\sim}\textbf{.}Sn)|$

$\geq \sum n \in \omega 1$

$= \infty)$

$\rightarrow 0)$

Accordingly we conclude

$\sim(A \in N)$

and the desired conclusion is at hand.

Lemma

7.74 $(\varphi \in \text{topaddor} \wedge \theta = \text{dor+}\,\varphi \wedge A \in \text{dmn}\,\varphi \rightarrow 0 \leq \textbf{.}\theta A < \infty)$

Proof:

Assume instead that

$(\textbf{.}\theta A = \infty)$,

and let

$(F = \text{E}\,\beta \subset A(\textbf{.}\varphi\beta \geq 1))$.

Notice that

$(\beta \in F$

$\rightarrow \textbf{.}\theta(A{\sim}\beta) = \infty$

$\rightarrow \bigvee \beta' \supset \beta(\beta'{\sim}\beta \in F \wedge \beta' \in F))$.

Choose then

$(S \in \text{sqnc}{\subset} F)$

so that

$\bigwedge n \in \omega(\textbf{.}S(n+1){\sim}\textbf{.}Sn \in F)$.

Now put

$(G = \bigvee n \in \omega\, \text{sng}(\textbf{.}S(n+1){\sim}\textbf{.}Sn))$

and note that

$(G \in \text{dsjn} \cap \text{sb}\, F \wedge \text{pwr}\, G = \omega)$

and according to 7.66.2,

$(\infty > \sum \beta \in G\,\textbf{.}\varphi(A\beta) = \sum \beta \in G\,\textbf{.}\varphi\beta \geq \sum \beta \in G 1 = \infty)$.

Helped by 7.66, 7.72, and 7.74 we infer

7.75 Theorems

.0 $(\varphi in\, \text{addor Upon sb}\, A\, \text{To rl} \wedge \textbf{.}\varphi A < \infty \rightarrow \varphi \in \text{topaddor})$

.1 $(\varphi \in \text{topaddor} \wedge \theta = \text{dor+}\,\varphi \rightarrow \theta \in \text{addor gauge On dmn}\,\varphi\, \text{To rf})$

7.76 Lemmas

.0 $(\varphi \in \text{addor Upon sb } A \text{ To rl} \wedge .\varphi A < \infty \wedge \psi = \text{dor}\pm\,\varphi \rightarrow .\psi A = .\varphi A)$

Proof:

Let

$(\varphi_1 = \text{dor}+\,\varphi \wedge \varphi_2 = \text{dor}-\,\varphi)$.

Because of 7.75

$(0 \leq .\varphi_1 A < \infty)$.

Also

$(\alpha \in \text{dmn}\,\varphi \rightarrow$
$.\varphi\alpha \leq .\varphi_1 A < \infty \wedge$
$.\varphi A$
$= .\varphi\alpha + .\varphi(A\sim\alpha)$
$\geq .\varphi\alpha -\ .\varphi_2 A \wedge$
$.\varphi A$
$= .\varphi(A\sim\alpha) + .\varphi\alpha$
$\leq .\varphi_1 A -\ -.\varphi\alpha)$.

$(.\varphi A \geq .\varphi_1 A -\ .\varphi_2 A \wedge$
$.\varphi A \leq .\varphi_1 A -\ .\varphi_2 A \wedge$
$.\varphi A = .\varphi_1 A + .\varphi_2 A = .\psi A)$

Applying .0 to $-\varphi$ we learn

.1 $(\varphi \in \text{addor Upon sb } A \text{ To rl} \wedge -\infty < .\varphi A \wedge \psi = \text{dor}\pm\,\varphi \rightarrow .\psi A = .\varphi A)$

From .0 and .1 we learn

.2 $(\varphi \in \text{addor Upon sb } A \text{ To rl} \wedge A \in \text{dmn}\,\varphi \wedge \psi = \text{dor}\pm\,\varphi \rightarrow .\psi A = .\varphi A)$

Theorem

7.77 $(\varphi \in \text{To rl} \wedge A \in \text{additive}''\,\varphi \wedge \psi = \text{dor}\pm\,\varphi \rightarrow .\psi A = .\varphi A)$

Proof:

Let

$(\mu = \text{strc}\,\varphi\,\text{sb}\,A \wedge \varphi_1 = \text{dor}+\,\varphi \wedge \varphi_2 = \text{dor}-\,\varphi \wedge \mu_1 = \text{dor}+\,\mu \wedge \mu_2 = \text{dor}-\,\mu)$.

With the help of 7.70.0 we learn

$(\mu \in \text{addor Upon sb } A \wedge A \in \text{dmn}\,\mu)$.

Since evidently

$(.\varphi A = .\mu A \wedge .\varphi_1 A = .\mu_1 A \wedge .\varphi_2 A = .\mu_2 A)$,

we may use 7.76.2 to conclude

$(.\varphi A = .\mu A = .\mu_1 A -\ .\mu_2 A = .\varphi_1 A -\ .\varphi_2 A = .\psi A)$.

We now have at once

The Jordan Decomposition Theorem

7.78 $(\varphi \in \text{addor To rl} \rightarrow \varphi = \text{dor}\pm\,\varphi)$

Theorem

7.79 $(\varphi \in \text{addor} \wedge \varphi' = \text{dor}'\,\varphi \wedge \varphi'' = \text{dor}''\,\varphi \rightarrow$

$\varphi' \in \text{addor On dmn}\,\varphi\,\text{To rl} \wedge$

$\varphi'' \in \text{addor On dmn}\,\varphi\,\text{To rl} \wedge$

$\varphi = \varphi' + \text{i}\cdot\varphi'')$

Proof:

It is easy to check that

$(\varphi' \in \text{addor On dmn}\,\varphi\,\text{To rl})$.

Now let

$(\psi'' = \curlywedge\beta\,\text{prt}''\,\boldsymbol{.}\,\varphi\beta)$

and note, with the help of 7.66, 7.75.1, and 7.69.1,

$(\psi'' \in \text{topaddor} \wedge -\psi'' \in \text{topaddor} \wedge \varphi'' \in \text{addor On dmn}\,\varphi\,\text{To rf})$.

Because of 7.77

.0 $(A \in \text{dmn}'\,\varphi'$

$\rightarrow A \in \text{additive}''\,\psi''$

$\rightarrow \boldsymbol{.}\,\varphi''A = \boldsymbol{.}\,\psi''A$

$\rightarrow \varphi A = \boldsymbol{.}\,\varphi'A + \text{i}\cdot\boldsymbol{.}\,\varphi''A)$.

On the other hand

.1 $(\boldsymbol{.}\,\varphi'A = \infty \vee \boldsymbol{.}\,\varphi'A = -\infty \rightarrow \boldsymbol{.}\,\varphi A = \boldsymbol{.}\,\varphi'A = \boldsymbol{.}\,\varphi'A + \text{i}\cdot\boldsymbol{.}\,\varphi''A)$.

The desired conclusion follows from .0 and .1.

7.80 Theorems

.0 $(\varphi \in \text{gauge} \wedge \text{dmn}\,\varphi \ni A \subset B \in \text{additive}''\,\varphi \rightarrow \boldsymbol{.}\,\varphi A \leq \boldsymbol{.}\,\varphi B)$

.1 $(\varphi \in \text{gauge} \wedge A \in \text{additive}''\,\varphi \wedge F \in \text{cbl sb dmn}\,\varphi \rightarrow \boldsymbol{.}\,\varphi(A \mathbin{\bar\nabla} F) \leq \sum\beta \in F\,\boldsymbol{.}\,\varphi(A\beta))$

Proof:

Let

$(M = \text{dmn}\,\varphi\,\text{sb}\,A \wedge \xi \in \text{On}\,F\,\text{Uto}\,\omega \wedge L = \curlywedge x \in F \sqsubseteq y(\boldsymbol{.}\,\xi y \in \boldsymbol{.}\,\xi x) \wedge$

$\bigwedge x \in F(\underline{\text{u}}x = x\!\sim \nabla\,\boldsymbol{.}\,Lx))$

and note that we may use 3.18A and .0 in checking

$(\boldsymbol{.}\,\varphi(A \mathbin{\bar\nabla} F)$

$= \boldsymbol{.}\,\varphi(A \bigvee x \in F\underline{\text{u}}x)$

$= \boldsymbol{.}\,\varphi \bigvee x \in F(A\underline{\text{u}}x)$

$= \sum x \in F\,\boldsymbol{.}\,\varphi(A\underline{\text{u}}x)$

$\leq \sum x \in F\,\boldsymbol{.}\,\varphi(Ax)$

$= \sum \beta \in F\,\boldsymbol{.}\,\varphi(A\beta))$.

.2 $(\varphi \in \text{addor gauge} \wedge \psi = \text{mr}\,\varphi \rightarrow \varphi \subset \psi \in \text{Msr rlm}\,\varphi \wedge \text{dmn}\,\varphi\,\text{mbl}\,\psi)$

Lemma

7.81 $(\varphi \in \text{addor To rl} \wedge \boldsymbol{.}\,\varphi A < \infty \wedge \theta = \text{vrn}\,\varphi$

$\rightarrow \bigvee P \subset A \bigwedge \alpha \in \text{dmn}\,\varphi \cap \text{sb}\,A(\boldsymbol{.}\,\theta\alpha = \boldsymbol{.}\,\varphi(\alpha P)))$

Proof:

Since according to 7.72 and 7.78

$(0 \leq \boldsymbol{.}\,\theta A < \infty)$

we can and do so choose

$(B \in \text{sqnc}\subset\text{sb}\,A)$

that

$\bigwedge n \in \omega(\boldsymbol{.}\,\theta A \leq \boldsymbol{.}\,\varphi\,\boldsymbol{.}\,Bn + \underline{2}\,n)$.

Now let

$$(C = \curlywedge n \in \omega \bigwedge m \in \omega \backsim n \,\mathbf{.}\, Bm \wedge P = \bigvee n \in \omega \,\mathbf{.}\, Cn)\ .$$

Clearly

$$\begin{aligned}
&(n \in \omega \rightarrow \\
&\quad \mathbf{.}\,\varphi(\mathbf{.}\,Bn \sim \mathbf{.}\,Cn) \\
&\qquad \leq \mathbf{.}\,\theta(\mathbf{.}\,Bn \sim \mathbf{.}\,Cn) \\
&\qquad = \mathbf{.}\,\theta(\mathbf{.}\,Bn \sim \bigwedge m \in \omega \backsim n \,\mathbf{.}\, Bm) \\
&\qquad \leq \mathbf{.}\,\theta(A \sim \bigwedge m \in \omega \backsim n \,\mathbf{.}\, Bm) \\
&\qquad = \mathbf{.}\,\theta(A \bigvee m \in \omega \backsim n \sim \mathbf{.}\, Bm) \\
&\qquad = \mathbf{.}\,\theta \bigvee m \in \omega \backsim n (A \sim \mathbf{.}\, Bm) \\
&\qquad \leq \sum m \in \omega \backsim n \,\mathbf{.}\, \theta(A \sim \mathbf{.}\, Bm) \\
&\qquad \leq \sum m \in \omega \backsim n \,\underline{2}\, m \\
&\qquad = 2 \cdot \underline{2}\, n)
\end{aligned}$$

Accordingly

$$\begin{aligned}
&(n \in \omega \rightarrow \\
&\quad \mathbf{.}\,\theta A \\
&\qquad \leq \mathbf{.}\,\varphi \,\mathbf{.}\, Bn + \underline{2}\, n \\
&\qquad = \mathbf{.}\,\varphi \,\mathbf{.}\, Cn + \mathbf{.}\,\varphi(\mathbf{.}\,Bn \sim \mathbf{.}\,Cn) + \underline{2}\, n \\
&\qquad \leq \mathbf{.}\,\varphi \,\mathbf{.}\, Cn + 2 \cdot \underline{2}\, n + \underline{2}\, n \\
&\qquad = \mathbf{.}\,\varphi \,\mathbf{.}\, Cn + 3 \cdot \underline{2}\, n)\ .
\end{aligned}$$

Hence

$$(\mathbf{.}\,\theta A \leq \operatorname{lin} n \,\mathbf{.}\, \varphi \,\mathbf{.}\, Cn + 0 = \mathbf{.}\,\varphi P \leq \mathbf{.}\,\theta A)$$

and

$$(\mathbf{.}\,\theta A = \mathbf{.}\,\varphi P)\ .$$

But

$$\begin{aligned}
&(\alpha \in \operatorname{dmn} \varphi \cap \operatorname{sb} A \\
&\quad \rightarrow 0 \\
&\qquad \leq \mathbf{.}\,\theta\alpha \;—\; \mathbf{.}\,\varphi(\alpha P) \\
&\qquad \leq \mathbf{.}\,\theta\alpha \;—\; \mathbf{.}\,\varphi(P\alpha) + \mathbf{.}\,\theta(A \sim \alpha) \;—\; \mathbf{.}\,\varphi(P \sim \alpha) \\
&\qquad = \mathbf{.}\,\theta A \;—\; \mathbf{.}\,\varphi P \\
&\qquad = 0 \\
&\rightarrow \mathbf{.}\,\theta\alpha = \mathbf{.}\,\varphi(\alpha P))\ .
\end{aligned}$$

Theorem

7.82 $(\varphi \in \operatorname{addor} \operatorname{To} \operatorname{rl} \wedge A \in \operatorname{dmn} \varphi \wedge \theta_1 = \operatorname{dor+} \varphi \wedge \theta_2 = \operatorname{dor} — \varphi$
$\rightarrow \bigvee P \subset A \bigwedge \alpha \in \operatorname{dmn} \varphi \cap \operatorname{sb} A(\mathbf{.}\,\theta_1 \alpha = \mathbf{.}\,\varphi(\alpha P) \wedge \mathbf{.}\,\theta_2 \alpha = \mathbf{.}\,\varphi(\alpha \sim P)))$

The Hahn Decomposition Theorem

7.83 $(\varphi \in \operatorname{addor}'' \operatorname{To} \operatorname{rl} \rightarrow \bigvee P \subset \operatorname{rlm} \varphi(\operatorname{dor+} \varphi = \operatorname{sct} \varphi P \wedge \operatorname{dor} — \varphi = \operatorname{sct} \varphi \sim P))$

7.84 Theorems

.0 $(\operatorname{addor} \subset \operatorname{gridor})$

.1 $(\varphi \in \operatorname{addor} \rightarrow \operatorname{vrn} \varphi \in \operatorname{addor})$

.2 $(\varphi \in \operatorname{addor} \rightarrow \operatorname{vrn} \varphi \subset \operatorname{var} \varphi \in \operatorname{Msr} \operatorname{rlm} \varphi \wedge \operatorname{dmn} \varphi \subset \operatorname{mbl} \operatorname{var} \varphi)$

Theorem
7.85 $(\varphi \in \operatorname{addor} \operatorname{To} \operatorname{rl} \rightarrow \operatorname{vrn} \varphi = \operatorname{dor+} \varphi + \operatorname{dor—} \varphi)$

We shall never use 7.86 and 7.87.
Theorem
7.86 $(\varphi \in \operatorname{addor} \operatorname{To} \operatorname{rl} \wedge P \subset \operatorname{rlm} \varphi \wedge P' \subset \operatorname{rlm} \varphi \wedge \theta = \operatorname{vrn} \varphi \wedge A \in \operatorname{dmn} \varphi \wedge$
$\operatorname{dor+} \varphi = \operatorname{sct} \varphi P = \operatorname{sct} \varphi P' \wedge \operatorname{dor—} \varphi = \operatorname{sct} \varphi {\sim} P = \operatorname{sct} \varphi {\sim} P'$
$\rightarrow . \theta(AP{\sim}P' \cup AP'{\sim}P) = 0)$

7.87 $(\varphi \in \operatorname{addor} \operatorname{To} \operatorname{rl} \wedge P \subset \operatorname{rlm} \varphi \wedge P' \subset \operatorname{rlm} \varphi \wedge \theta = \operatorname{var} \varphi \wedge$
$\operatorname{dor+} \varphi = \operatorname{sct} \varphi P = \operatorname{sct} \varphi P' \wedge \operatorname{dor—} \varphi = \operatorname{sct} \varphi {\sim} P = \operatorname{sct} \varphi {\sim} P'$
$\rightarrow . \theta(P{\sim}P' \cup P'{\sim}P) = 0)$

Special Integrals

7.88 Definitions

.0 (interval $\equiv$ ⋿ $A \in$ sb rf ∽1 $\bigwedge x, y, \in A \bigwedge \lambda \in$ nt 01$((1 — \lambda) \cdot x + \lambda \cdot y \in A))$

.1 (ntf $f \equiv$ (sng$(0, 0) \cup$ ⋌ $\beta \in$ interval(. f Sup β — . f Inf β)))

Definition

7.89 ($\int ab\underline{u}x\varphi\,\mathrm{d}\,x \equiv ((a \le b \to \int$ nt $ab; \underline{u}x\varphi\,\mathrm{d}\,x) \wedge (b < a \to \int$ nt $ab; \underline{u}x\varphi\,\mathrm{d}\,x)))$

7.90 Definitions

.0 ($\int \underline{u}x\,\mathrm{d}\,x \equiv \int \underline{u}x\,\mathcal{L}\,\mathrm{d}\,x$)

.1 ($\int A; \underline{u}x\,\mathrm{d}\,x \equiv \int A; \underline{u}x\,\mathcal{L}\,\mathrm{d}\,x$)

.2 ($\int ab\underline{u}x\,\mathrm{d}\,x \equiv \int ab\underline{u}x\,\mathcal{L}\,\mathrm{d}\,x$)

7.91 Definitions

.0 ($\int \underline{u}xg\,\partial\,x \equiv \int \underline{u}x$ ntf $g\,\mathrm{d}\,x$)

.1 ($\int \underline{u}xg$ dil $x \equiv \int \underline{u}xg\,\partial\,x$)

.2 ($\int A; \underline{u}xg\,\partial\,x \equiv \int A; \underline{u}x$ ntf $g\,\mathrm{d}\,x$)

.3 ($\int ab\underline{u}xg\,\partial\,x \equiv \int ab\underline{u}x$ ntf $g\,\mathrm{d}\,x$)

7.92 Definitions

.0 ($\int \underline{u}x\,\partial\,x \equiv \int \underline{u}x$ ⋌ $xx\,\partial\,x$)

.1 ($\int A; \underline{u}x\,\partial\,x \equiv \int A; \underline{u}x$ ⋌ $xx\,\partial\,x$)

.2 ($\int ab\underline{u}x\,\partial\,x \equiv \int ab\underline{u}x$ ⋌ $xx\,\partial\,x$)

7.93 Theorems

.0 (a ∽$\in$ rl $\vee$ b ∽$\in$ rl $\to \int ab\underline{u}x\varphi\,\mathrm{d}\,x =$ U)

.1 ($a < b \to \int ab\underline{u}x\varphi\,\mathrm{d}\,x = -\int ba\underline{u}x\varphi\,\mathrm{d}\,x$)

It can be shown that

$(-\infty < a < b < \infty \wedge$

kf $\ni R =$ The Riemann−Stieltjes integral from a to b of f with respect to g

$\to \int ab\,.\,fxg\,\partial\,x = R)$.

With integration by parts in mind we ask when does

$(\int ab\,.\,fxg\,\partial\,x = .\,fb \cdot .\,gb — .\,fa \cdot .\,ga — \int ab\,.\,gxf\,\partial\,x)$?

In 7.92 we have an interesting generalization of the Riemann integral. In particular

$(\int 0\infty(1/(1 + x \cdot x))\,\partial\,x = \pi/2)$.

Measure and Integral Size

Theorem
7.94 $(\varphi \in \mathrm{gauge} \wedge f \in \mathrm{gauge} \wedge \theta = \mathrm{mr}\,\varphi \wedge \int \boldsymbol{.}\, fx\varphi \,\mathrm{d}x \leq \lambda \rightarrow \boldsymbol{.}\,\theta \mathsf{E}\, x \in \mathrm{rlm}\,\varphi(\boldsymbol{.}\, fx \geq 1) \leq \lambda)$

Proof:

Let

$(r > 0 \wedge M = \mathrm{mode}\,\varphi \wedge A = \mathsf{E}\, x \in \mathrm{rlm}\,\varphi(\boldsymbol{.}\, fx \geq 1))$

and choose

$(D \in \mathrm{grid}\,\varphi)$

so that

$\bigwedge \xi \in \mathrm{vs}\, MD(\mathrm{rsum}\, f\xi\varphi \leq \lambda + r)$.

Next let

$(D' = \mathsf{E}\, \beta \in D(\beta A \neq 0))$

and so choose

$(\eta \in \mathrm{swath}\, D)$

that

$\bigwedge \beta \in D'(\boldsymbol{.}\,\eta\beta \in A)$,

and infer

$(A \subset \nabla D \wedge$

$\lambda + r$

$\geq \sum \beta \in D(\boldsymbol{.}\, f \boldsymbol{.}\,\eta\beta \bullet \boldsymbol{.}\,\varphi\beta)$

$\geq \sum \beta \in D'(\boldsymbol{.}\, f \boldsymbol{.}\,\eta\beta \bullet \boldsymbol{.}\,\varphi\beta)$

$\geq \sum \beta \in D' \boldsymbol{.}\,\varphi\beta$

$\geq \boldsymbol{.}\,\theta A)$

Recall now the arbitrary nature of r.

It is now easy to check

Theorem
7.95 $(\varphi \in \mathrm{gauge} \wedge f \in \mathrm{gauge} \wedge \theta = \mathrm{mr}\,\varphi \wedge \int \boldsymbol{.}\, fx\varphi \,\mathrm{d}x = 0 \rightarrow \mathsf{E}\, x \in \mathrm{rlm}\,\varphi(\boldsymbol{.}\, fx > 0) \in \mathrm{zr}\,\theta)$

Seminar Integration: Word Index

This index is to the symbols made up primarily as letter combinations. They are listed alphabetically with a reference to their first location in the Seminar Integration notes. "BN" references are to Background Notation.

Seminar Integration: Symbol Index

This index is to the non-word symbols. They are listed in the order of their first appearance in the Seminar Integration notes. "BN" references are to Background Notation.

Symbol	Location	Meaning
(	BN.0, 1.0.0	left parenthesis
$\wedge$	BN.0, 1.1.0	and
$\leftrightarrow$	BN.0	if and only if
)	BN.0, 1.0.0	right parenthesis
$\vee$	BN.1	or
$\sim$	BN.2	not / complement
$\bigwedge$	BN.3, 1.1.0	for each / indexed intersection
,	BN.3	comma / ordered pair
$\bigvee$	BN.4	for some / indexed union
$\in$	BN.5, 1.1.0	belongs to, is a member of
$\rightarrow$	BN.5	implies
$\ni$	BN.7	holds, has as a member
∽	BN.8	set difference, set minus
$\subset$	BN.9	is a subset of
$\subset\cdot$	BN.10	is a proper subset of
$\neq$	BN.10	not equal
$\cap$	BN.11	intersect (binary)
$=$	BN.11, 1.1.3	equals, is equal to
E	BN.15, 1.1.0	classifier, set of
$\{$	BN.15	left brace
:	BN.15, 1.18	colon, used in 1.18 for composition of relations $(R : S)$
$\}$	BN.15	right brace
$\triangledown$	BN.16	union of
$\sqcap$	BN.17, 1.1.0	intersection of
0	BN.19, 1.0.0	false / empty set, zero
$_*$	BN.34	image; $_*RA$
*	BN.35	inverse image; $^*RB$
$\centerdot$	BN.47	function value; $\centerdot fx$
⋏	BN.48	function builder ("lonzo")
ω	BN.51, 1.1.0	omega; set of natural numbers beginning with 0
1	BN.52, 1.0.0	one
2	BN.52, 1.0.1	two
3	BN.52, 1.0.2	three
$+$	BN.52, 1 1st sentence	plus (addition)
$\cup$	BN.52	union (binary)
$\vert$	BN.56, 1.14.2	absolute value bar; $\vert x \vert$
∞	BN.56, 1.1.18	infinity
$-$	BN.57, 1 1st sentence	negative
$\leq$	BN.57, 1.1.15	less than or equal to
$<$	BN.58, 1.1.14	less than
$\sum$	BN.60, 1.67.2	sum
$\surd$	BN.61, 1.12	square root
$\cdot$	BN.61, 1 1st sentence	times
/	BN.65, 1 1st sentence	divided by
$-$	BN.71	minus (subtraction)
$\underline{2}$	BN.80	1/2 to an integral power

$^{\bullet}$	BN.50	raise to an integral power; $^{\bullet}xn$
$\geq$	BN.87	greater than or equal to
Sup	1.1st sentence	supremum of a set
i	1.1st sentence	imaginary unit, square root of $Neg1$
$\equiv$	1.0.0	definor symbol
scsr	1.0.0	successor
4	1.0.3	four
5	1.0.4	five
6	1.0.5	six
7	1.0.6	seven
8	1.0.7	eight
9	1.0.8	nine
$\infty\!\!\!\!\mid$	1.1.7	complex infinity
$>$	1.1.16	greater than
$\geq$	1.1.17	greater than or equal to
ω'	1.7.0	omega prime; the set of integers
$\cdot\!\rightarrow$	1.23.2runs along	
$\hookrightarrow$	1.23.3	slides to
$\bullet$	1.87	heavy times or big dot
$\sum$	1.136	symmetric sum
$\int$	7.0.1	integral
#	7.0.1	pound sign, used in preliminary integral
$\underline{\}}$	7.0.1	right brace, used in preliminary integral
$\overline{\int}$	7.0.2	upper integral
$\underline{\int}$	7.0.3	lower integral
$\overline{]}$	7.8.2	right bracket, used in preliminary integral
d	7.8.3	integral constant, bound variable indicator
;	7.8.4	such that
$\underline{\cap}$	7.18.4	common refinement
∇meet	7.65A	sigma-meet
∂	7.91.0	dil, integral constant, bound variable indicator