Facet Exchange Groups

Bob Neveln

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Definitions

An *abstract complex*, $(X, \leq, \dim)$, is a set of cells X with an ordering $\leq$ and an assignment dim of a dimension $\mathbf{n} \in \mathbf{N} \cup \{-1\}$ to each $\mathbf{x} \in \mathbf{X}$ such that

1. $\mathbf{x} \leq \mathbf{y} \rightarrow \dim \mathbf{x} \leq \dim \mathbf{y}$
2. $\mathbf{x} \leq \mathbf{y}\ \ \&\ \ \dim \mathbf{x} = \dim \mathbf{y} \rightarrow \mathbf{x} = \mathbf{y}$

We further assume that if $\dim \mathbf{x} \geq 0$ then there exists a set of boundary cells $\mathbf{B}(\mathbf{x}) \subset \mathbf{X}$ such that

1. For all $\mathbf{y} \in \mathbf{B}(\mathbf{x})$, $\mathbf{y} < \mathbf{x}$ and $\dim \mathbf{y} + 1 = \dim \mathbf{x}$.
2. For all $\mathbf{z} \in \mathbf{X}$, if $\mathbf{z} < \mathbf{x}$ then for some $\mathbf{y} \in \mathbf{B}(\mathbf{x}), \mathbf{z} \leq \mathbf{y}$.
3. $\mathbf{B}(\mathbf{x}) \neq \emptyset$.

By a *ground cell* of $\mathbf{X}$ we mean a cell which is not a boundary cell, i.e. $\mathbf{x}$ is a ground cell of $\mathbf{X}$ if and only if $\mathbf{x} \in \mathbf{X}$ and for no $\mathbf{y} \in \mathbf{x}$ is $\mathbf{x} < \mathbf{y}$.

A complex $\mathbf{X}$ is *homogeneous of dimension* $\mathbf{n}$ if and only if

1. For every $\mathbf{x} \in \mathbf{X}$ there is a $\mathbf{y}$ such that $\mathbf{y}$ is a ground cell of $\mathbf{X}$ and $\mathbf{x} < \mathbf{y}$.
2. Every ground cell of $\mathbf{X}$ has dimension $\mathbf{n}$
3. $\mathbf{X} \neq \emptyset$.

A homogeneous complex $\mathbf{X}$ is *tilewise connected* if for any ground cells $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ there is a sequence of ground cells $\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_\mathbf{n}$ such that $\mathbf{x} = \mathbf{z}_0 \;\&\; \mathbf{y} = \mathbf{z}_\mathbf{n} \;\&\; \mathbf{B}(\mathbf{z}_\mathbf{k}) \cap \mathbf{B}(\mathbf{z}_{\mathbf{k}+1}) \neq \emptyset$ for all $\mathbf{k}$, $0 \leq \mathbf{k} < \mathbf{n}$.

A homogeneous complex $\mathbf{X}$ of dimension $\mathbf{n}$ is *uncrowded* if and only if for each $\mathbf{y} \in \mathbf{X}$, if $\dim \mathbf{y} = \mathbf{n} - 1$ then there exist at most two $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{y} \in \mathbf{B}(\mathbf{x})$.

By a *paved* complex we mean a homogeneous complex which is tilewise connected and uncrowded.

In an uncrowded complex of dimension $\mathbf{n}$, for each cell $\mathbf{y}$ of dimension $(\mathbf{n}-1)$ the number of cells $\mathbf{x}$ such that $\mathbf{y} \in \mathbf{B}(\mathbf{x})$ is either 1 or 2.

By the *boundary* of a homogeneous complex $\mathbf{X}$ of dimension $\mathbf{n}$ we mean the set of cells $\mathbf{y}$ for which there is exactly one $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{y} \in \mathbf{B}(\mathbf{x})$.

By an *closed* complex we mean a complex whose boundary is empty.

By a *polyhedral* complex $\mathbf{X}$ we mean a paved complex in which for every $\mathbf{x} \in \mathbf{X}$ if $\dim \mathbf{x} \geq 0$ then $\mathbf{B}(\mathbf{x})$ is a closed paved complex.

Given a polyhedral complex $\mathbf{X}$ of dimension $\mathbf{n}$ we define a *choice sequence* as a sequence of cells $\mathbf{x_n}, \mathbf{x_{n-1}}, \ldots, \mathbf{x}_0$ such that $\mathbf{x_k} \in \mathbf{B}(\mathbf{x_{k+1}})$ for $\mathbf{k} < \mathbf{n}$.

The foregoing definitions, with minor alterations, were taken from a lengthy investigation of topology, done jointly with Bob Alps.

Choice Sequences of a Cube

1. Sequence Format: Face, Edge, Vertex

2. Cells
   - 6 Faces: Top, Bottom, Front, Rear, Right,Left
   - 12 Edges: Top-Front, Top-Right, Top-Rear, etc.
   - 8 Vertices: Top-Front-Right, Bottom-Left-Rear, etc.

3. Two Examples of a Choice Sequence
   - Top, Top-Right, Top-Right-Front
   - Front, Front-Left, Front-Left-Bottom

4. There are $6 \times 4 \times 2 = 48$ such choice sequences

The Facet Exchange Group

- Given a closed polyhedral complex X of dimension n and a dimension m, $0 \leq m \leq n$, then for each choice sequence x there is exactly one choice sequence y, such that
  1. $y_k \neq x_k$ if $k = m$
  2. $y_k = x_k$ if $k \neq m$
- If we map each choice sequence $x \mapsto y$, as determined above, this defines a function F_m on the set C of all choice sequences on X

$$F_m : C \to C$$

- F_m is a permutation of C consisting of 2-cycles.
- The *facet exchange group* of a closed polyhedral complex X of dimension n is the permutation group generated by the permutations $\langle F_0, F_1, \ldots, F_n \rangle$.
- The generators $F_0, F_1, \ldots, F_n$ are called *flips*. There is one flip for each dimension.

Facet Exchange Group of the Cube

- For the cube there are three flips, F (face) E (edge) and V (vertex)

$$F : C \to C \qquad E : C \to C, \qquad V : C \to C$$

- Because the flips consist of 2-cycles they satisfy the relations

$$F^2 = E^2 = V^2 = 1$$

- But FE rotates around vertices; EV cycles around the square faces and F commutes with V so they also satisfy the relations:

$$(FE)^3 = (EV)^4 = (FV)^2 = 1$$

- It turns out that these relations are sufficient to define the group.
- It is therefore a Coxeter group. It has order 48 and is isomorphic to the usual symmetry group.

Can we Reconstruct the Complex from the Group?

- Given the permutation group and its generators? YES.
- Given only the abstract group? NO

Given $F_0, F_1, \ldots, F_n : C \to C$

- Any subgroup of $\langle F_0, F_1, \ldots, F_n \rangle$ partitions the set C into orbits.
- Define X_m as the set of orbits produced by the subgroup

$$\langle F_0, \ldots, F_{(m-1)}, F_{(m+1)}, \ldots, F_n \rangle$$

.

- We can recover the set of cells:

$$X = \bigcup_{m=0}^{n} X_m$$

- We can recover the dimension: given an orbit $x \in X$, $\dim x$ is the unique m such that $x \in X_m$.
- We can recover the ordering:

$$x \leq y \leftrightarrow x \cap y \neq \emptyset \;\; \& \;\; \dim x \leq \dim y$$

Given the Abstract Group $G = \langle F_0, \ldots, F_n \rangle$

- The group generated by the reversed sequence $F_n, \ldots, F_0$ is isomorphic to G. The abstract group cannot distinguish a complex from its dual, e.g. a cube from an octahedron.
- Complexes representing the Klein Bottle and the Projective Plane can yield identical groups.

Representing $G = \langle F_0, \ldots, F_n \rangle$ as a Semidirect Product

- Given $G = \langle F_n, \ldots, F_0 \rangle$ the subgroup $\langle F_{(n-1)}, \ldots, F_0 \rangle$ represents facet exchanges internal to the ground cell of each facet.
- In some cases there is a normal subgroup of $N \subset G$ such that G is an internal product:

$$G = N\langle F_{(n-1)}, \ldots, F_0 \rangle$$

- If $N \cap \langle F_{(n-1)}, \ldots, F_0 \rangle = \{1\}$ then this product is a semidirect product.
- Some of the simplest examples are infinite:
  - The Infinite Dihedral Group
  - A Tiling of the Plane into Rectangles

A Tiling of the Plane into Rectangles

- A group which tiles the plane is generated by F,E,V with relations:

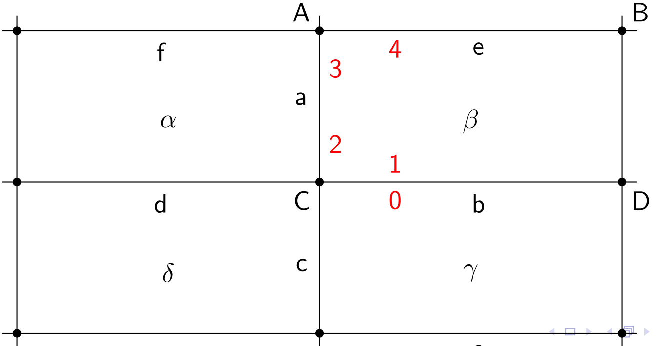
$$F^2 = E^2 = V^2 = (FE)^4 = (EV)^4 = (FV)^2 = 1$$

- There is a Normal Subgroup which is a free group on two generators

$$N = \langle FEVE, EFEV \rangle, \qquad N \cap \langle E, V \rangle = \{1\}$$

- The Factor Group, $\langle E, V \rangle$ is isormorphic to D_4.
- G is a semidirect product:

$$G = ND_4$$



The Torus Decomposed into Four Rectangles

- The torus can also be described as a semidirect product. As with the plane we have the relations:

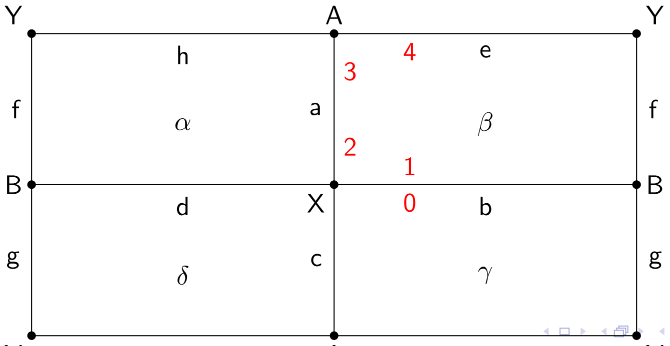
$$F^2 = E^2 = V^2 = (FE)^4 = (EV)^4 = (FV)^2 = 1$$

- Again there is a Normal Subgroup with two generators;

$$N = \langle FEVE, EFEV \rangle, \qquad N \cap \langle E, V \rangle = \{1\}, \qquad G = ND_4$$

- But here N is isomorphic to the Klein 4-Group, $C_2 \times C_2$ with:

$$(FEVE)^2 = (EFEV)^2 = ((FEVE)(EFEV))^2 = 1$$



The Cube is not a Semidirect Product

- We have the relations:

$$F^2 = E^2 = V^2 = (FE)^3 = (EV)^4 = (FV)^2 = 1$$

- There is a Normal Subgroup N (generated by all elements of order 3), such that:

$$G = ND_4$$

- But this N has order 24 and

$$N \cap \langle E, V \rangle = \{1, E, VEV, EVEV\}$$

- If we let N be the cyclic group $\langle FEV \rangle$ of order 6 we have that

$$G = ND_4 \qquad o(G) = o(N) \cdot o(D_4)$$

but $\langle FEV \rangle$ is not a normal subgroup.

The Klein Bottle Decomposed into Four Rectangles

- The Klein Bottle can tentatively be described as a semidirect product. As with the plane we have the relations:

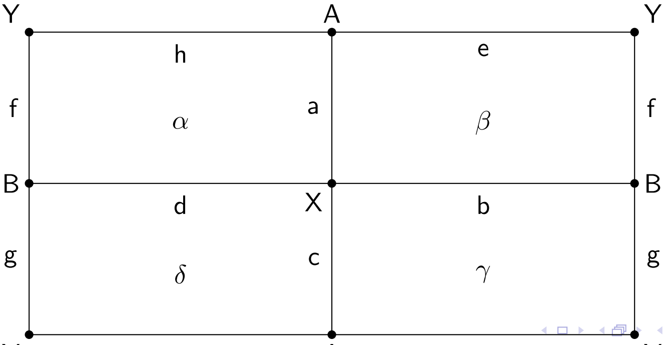
$$F^2 = E^2 = V^2 = (FE)^4 = (EV)^4 = (FV)^2 = 1$$

- Again there is a Normal Subgroup with two generators;

$$N = \langle FEVE, EFEV \rangle, \qquad N \cap \langle E, V \rangle = \{1\}, \qquad G = ND_4$$

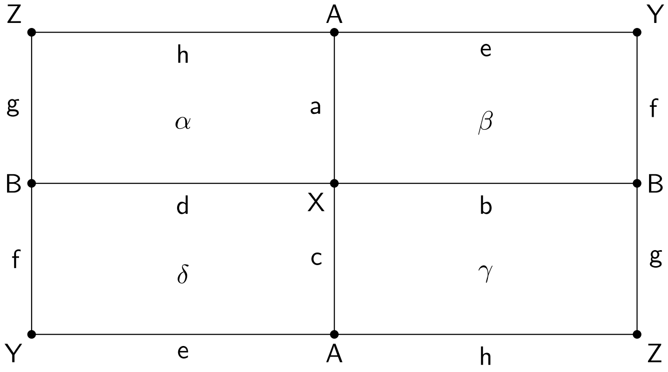
- But here N is isomorphic to $C_4 \times C_4$ with:

$$(FEVE)^4 = (EFEV)^4 = ((FEVE)(EFEV))^2 = 1$$



The Projective Plane Decomposed into Four Rectangles

- Again tentatively, this group seems to have the very same presentation as the preceding group.



Orientations:

- The set of all products of even length taken from the generating set $\{F_0, \ldots, F_n\}$ forms a subgroup. Call this subgroup E. E is either a proper subgroup of index 2 or it is the whole group.
- An orientation of the complex exists if and only if E partitions C into two orbits.
- Question: Is E *always* a proper subgroup of $\langle F_0, \ldots, F_n \rangle$?

Does the group tell us anything the symmetry of the complex?

- The size of the group increases as symmetry decreases. The regular solids have groups which are Coxeter groups isomorphic to their usual symmetry groups.
- Complexes lacking in symmetry have much larger groups.

Facet Exchange Group of the Square Based Pyramid

- Facets
  - 5 Faces: Bottom and 4 Sides
  - 8 Edges: 4 Bottom Edges and 4 Edges Slanting to the Top
  - 5 Vertices: 4 Bottom Corners and the Top
- As with the cube we have the relations: $F^2 = E^2 = V^2 = 1$. Also as with the cube FE rotates around vertices and EV cycles around the faces. There are vertices where 3 faces meet and one where 4 faces meet; and their least common multiple is 12. There are faces with three sides ond one with 4 and again the least common multiple is 12. These considerations show that the following relations are satisfied:
$$(FE)^{12} = (EV)^{12} = (FV)^2 = 1$$
- If these were the only relations the group would not be finite. But as a permutation group on a finite set it is finite and so is is not a Coxeter group.
- The group can be calculated. Its order is $6144 = 3 \times 2^{11}$.
- Since $\langle E, V \rangle$ is isomorphic to D_{12} we seek a group N of order 256 such that
$$G = N\langle E, V \rangle$$

Questions

- What distinguishes complexes whose facet exchange groups can be written as a semidirect product from those which cannot?
- Which properties of a complex can be determined from the abstract group as opposed to the permutation group?
- Does the decomposition of a closed complex into open complexes have anything to say about a decomposition of groups into pieces which would not be groups?

Help Wanted

I'd like to work with anyone interested in this problem!